Math 301 A - Spring 2014
Midterm Exam
April 23, 2014
Answers

1. *Find the gcd of $a = 163438$ and $b = 16150$ and express the gcd as a linear combination of a and b.*

   Applying the Euclidean algorithm, we have

$$\begin{aligned} 163438 &= (10)(16150) + 1938 && \text{so } (163438, 16150) = (16150, 1938); \\ 16150 &= (8)(1938) + 646 && \text{so } (163438, 16150) = (1938, 646); \\ 1938 &= (3)(646) + 0 && \text{so } (163438, 16150) = 646. \end{aligned}$$

   Using these equations, we can write

$$\begin{aligned} 646 &= 16150 - (8)(1938) \\ 646 &= 16150 - (8)(163438 - (10)(16150)) \\ 646 &= (81)(16150) - (8)(163438). \end{aligned}$$

2. *How many zeros does $321!$ end in (when written in decimal notation)?*

   We know that, for any prime, and any positive integer n,

$$\mathrm{ord}_p n! = \sum_{i=1}^{\infty} \left[\frac{n}{p^i}\right]$$

   where only finitely many terms are non-zero: once $p^i > n$ then $\left[\frac{n}{p^i}\right] = 0$.

   The number of trailing of zeros of $321!$ is the value of k such that 10^k exactly divides $321!$.

   Since $10 = 5 \cdot 2$, this k value can be determined by finding $\mathrm{ord}_2 321!$ and $\mathrm{ord}_5 321!$.

   However, since $\left[\frac{n}{2^i}\right] \geq \left[\frac{n}{5^i}\right]$ for all $i > 0$, we can simply find $\mathrm{ord}_5 321!$:

$$\mathrm{ord}_5 321! = \left[\frac{321}{5}\right] + \left[\frac{321}{5^2}\right] + \left[\frac{321}{5^3}\right] = \left[\frac{321}{5}\right] + \left[\frac{321}{25}\right] + \left[\frac{321}{125}\right] = 64 + 12 + 2 = 78.$$

   Hence, $5^{78} || 321!$, and so 10^{78} divides $321!$, but 10^{79} does not divide $321!$.

   Thus, $321!$ ends in 78 zeros. If we like, we can check this by counting them: $321! =$
   679269174457380047028785170185919186947307915378873794717504834800056699
   620107556588363406711769787197195178862008179089783397511787291509841159
   447296698243478466739056566182553499706936922318110750836973692573813672
   250633204183009258104385355180663770974611994543043088808911065034505710
   742622493294337180339627744007411619661923211692633961412869634120499252
   010840025650326123715557128540459760461684735762027568521406316170120640
   288596098543945942754314954146518452656699065041569649506333465354135988
   135665347667173854434722462264095651475843741418032851023524292353047920
   605881853542000000000000000000000000000000000000000000000000000000000000
   000000000000000000.

3. *(a) Give an example of two different positive integers with (exactly) 600 divisors.*

We know that p^m, where p is a prime and m is a positive integer has exactly $m+1$ divisors.

So, 2^{599} and 3^{599} are two different positive integers with exactly 600 divisors.

(b) What is the smallest positive integer with (exactly) 600 divisors that you can find? (You do not have to find the smallest integer with 600 divisors, just try to find one as small as you can - the smaller the better!)

We know that if n is positive integer with prime factorization

$$n = p_1^{\alpha_1} \cdot p_k^{\alpha_k}$$

with all p prime and all α positive, then n has $(\alpha_1 + 1) \cdot (\alpha_k + 1)$ divisors.

So, if we want 600 divisors, we can factor 600 and match up the factors with these $\alpha + 1$ factors.

For example, $600 = 2 \cdot 300$. Subtracting 1 from 2 and 300 we have 1 and 299. Hence, the number $2 \cdot 3^{299}$ has 600 divisors.

But, $2^{299} \cdot 3$ has 600 divisors, too, and is smaller.

By factoring 600 further, we can get still smaller ones. For example, since $600 = 5 \cdot 5 \cdot 3 \cdot 2 \cdot 2 \cdot 2$, the number

$$32432400 = 2^4 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$$

has 600 divisors (this happens to be the smallest such number, but it takes a little work to prove this - the smallest number with m divisors is not always found by factoring m into primes and then just subtracting one from each prime and throwing them into exponents - try finding the smallest number with 64 divisors, for instance).

4. *For what integers n is $n^2 - 1$ a prime number? State and prove a theorem that answers this question.*

Theorem: Suppose n is an integer. Then $n^2 - 1$ is prime iff $n = \pm 2$.

Proof: Suppose $n = \pm 2$. Then $n^2 - 1 = 3$, and 3 is prime.

Now, suppose n is an integer and $p = n^2 - 1$ is prime.

If $n = 0$, then $n^2 - 1 = -1$ which is not prime.

If $n = \pm 1$, then $n^2 - 1 = 0$, which is not prime.

Since $(-n)^2 - 1 = n^2 - 1$, we will assume for now that $n > 1$.

Suppose $n > 2$. Then $n - 1 > 1$ and so $p = (n + 1)(n - 1)$ is a product of two integers greater than 1 and hence is not prime. This contradicts our assumption that p is prime.

Hence, $n \leq 2$, and so $n = 2$ is the only possible value for $n > 0$.

Hence if $n^2 - 1$ is prime, then $n = \pm 2$.

Thus, $n^2 - 1$ is prime iff $n = \pm 2$. ■

5. *Prove that the product of two consecutive integers, both greater than 2, has at least three (not necessarily distinct) prime factors.*

   Proof: Let $a < b$ be two consecutive integers, with $a > 2$.

   Then $b - a = 1$, so $(a, b) = 1$.

   Exactly one of a and b is even; without loss of generality, suppose a is even.

   Then $(a, b) = (\frac{a}{2}, b)$ and, since $a > 2$, $\frac{a}{2} > 1$.

   Hence, there exist (distinct) primes p_1 and p_2 such that $p_1 | \dfrac{a}{2}$ and $p_2 | b$.

   Hence, $\left(\dfrac{a}{2}\right) b$ has at least 2 prime factors, and so $2\left(\dfrac{a}{2}\right) b = ab$ has at least 3 prime factors.■

6. Prove that $\dfrac{\ln 2}{\ln 3}$ is irrational.

   Proof: Suppose $\dfrac{\ln 2}{\ln 3}$ is rational.

   Then there exist integers a and b, $b \neq 0$, with $\dfrac{\ln 2}{\ln 3} = \dfrac{a}{b}$.

   Then we have

$$\begin{aligned} b \ln 2 &= a \ln 3 \\ \ln 2^b &= \ln 3^a \\ 2^b &= 3^a. \end{aligned}$$

   As a and b are integers, there are a few things we can say at this point.

   By the Fundamental Theorem of Arithmetic, we know that two different factorizations of a positive integer is impossible. So this is a contradiction.

   Or, we might just note that, since $b \neq 0$, 2 divides 2^b and so must divide 3^a. Since we know $2 \not| \, 3^a$, this is a contradiction.

   Hence, we may conclude that our assumption that $\dfrac{\ln 2}{\ln 3}$ is rational is false.

   Thus, $\dfrac{\ln 2}{\ln 3}$ is irrational. ■