A short introduction to bases

(note: throughout, square brackets indicate the greatest integer of the enclosed quantity)

When we write the number 1251, we are expressing the quantity

$$1 \times 10^3 + 2 \times 10^2 + 5 \times 10^1 + 1 \times 10^0.$$

When we express a number in this way, we say that 10 is the **base**.

We can express 1251 in other bases.

For instance, suppose we use the base 12.

If we look at powers of 12, we see

$$\begin{aligned} 12^0 &= 1 \\ 12^1 &= 12 \\ 12^2 &= 144 \\ 12^3 &= 1728 \end{aligned}$$

Since $1728 > 1251$, we won't need it to represent 1251 in base 12.

Instead, we start by seeing how many multiples of 144 are needed:

$$\frac{1251}{12} = 8.6875$$

so $\left[\frac{1453}{12}\right] = 8$.

This tells us that the third base-12 digit of 1251 is 8. Then, we subtract: $1251 - (8)(144) = 99$. Dividing 99 by 12 we see that $\left[\frac{99}{12}\right] = 8$. Subtracting, we find $99 - (8)(12) = 3$. So the right-most digit is 3.

What we've done is calculate that

$$1251 = 8 \times 12^2 + 8 \times 12^1 + 3 \times 12^0$$

and so, in base 12, 1251 would be written as 883.

Notationally this can be confusing, so one convention is to subscript the numbers with the base to keep things straight:

$$1251_{10} = 883_{12}.$$

Formally, what we are doing is this. Given a base β that is a positive integer and a positive integer n, we are determing a finite sequence of integers $\{a_k, a_{k-1}, \ldots, a_1, a_0\}$, with $0 \leq a_i < \beta$ for $i = 0, \ldots, k-1$ and $1 \leq a_k < \beta$ such that

$$n = a_k\beta^k + a_{k-1}\beta^{k-1} + \cdots + a_1\beta + a_0.$$

For negative n, you can do the same thing, and just preface it all with a minus sign.

When it comes to writing out such an expression, we can run into some trouble if any of the a_i is greater than or equal to 10, since if we don't take special measures, the expression may be ambiguous. For instance, in base 12, if we write 3210, do we mean

$$3 \times 12^3 + 2 \times 12^2 + 1 \times 12^1 + 0$$

or do we mean

$$3^2 + 2 \times 12^1 + 10?$$

To get around this problem, for small bases anyway, some people use capital letters of the alphabet to represent integers greater than 9: A$=10$, B$=11$, C$=12$, etc. This is especially common in hexadecimal, i.e., base 16, where we have expressions like

$$\mathrm{A5C2}_{16} = 10 \times 16^3 + 5 \times 16^2 + 12 \times 16^1 + 2 = 42343_{10}$$

Here's another example of a base conversion. Suppose we want to write 578 in base 2 (also known as *binary*).

We can begin by considering powers of 2.

$$\begin{aligned} 2^0 &= 1 \\ 2^1 &= 2 \\ 2^2 &= 4 \\ 2^3 &= 8 \\ 2^4 &= 16 \\ 2^5 &= 32 \\ 2^6 &= 64 \\ 2^7 &= 128 \\ 2^8 &= 256 \\ 2^9 &= 512 \\ 2^{10} &= 1024 \end{aligned}$$

Since $2^{10} > 578$, we do not need it when writing 578 in binary. Binary is actually the easiest base to do conversions to, since the only digits are 0 and 1, so we just have to decide what is the largest power needed at each stage.

Since $578 > 512$, we need a 1 in the 2^9 place.

Then, $578 - 512 = 66$, so we have 0 at the 2^8 and 2^7 place, but a 1 at the 2^6 place.

Then $66 - 2^6 = 2$, so all other places have zeros, except for a 1 at the 2^1 place.

Thus, we know

$$578 = 1\times2^9+1\times2^6+1\times2^1 = 1\times2^9+0\times2^8+0\times2^7+1\times2^6+0\times2^5+0\times2^4+0\times2^3+0\times2^2+1\times2^1+0\times2^0$$

and so,

$$578_{10} = 1001000010_2$$

As you might guess, the smaller the base, the more digits you will need to represent a number. In fact, it is not hard to show that a positive integer n requires

$$1 + \left[\frac{\ln n}{\ln \beta} \right]$$

digits to be represented in base β.

Exercises

For practice, you can verify the following equivalencies. (You can do this without working through the conversion procedure illustrated in examples above, but that's probably what you ought to practice here, since the other way to verify these is relatively trivial.)

1. $123_{10} = 234_7 = 102_{11} = 11120_3$
2. $5000_{10} = 2378_{13} = 1735_{15}$
3. $1001110_2 = 58_{14}$
4. $F4240_{16} = 1000000_{10}$