

Math 300 E - Winter 2019
Midterm Exam Number One
Answers
January 30, 2019

1. Let a and b be integers. Write useful contrapositives of the following statements.

(a) If ab is even, then a is even or b is even.

If a and b are odd, then ab is odd.

(b) If $a > 0$ and $b > 0$, then $ab > 0$ and $a + b > 0$.

If $ab \leq 0$ or $a + b \leq 0$, then $a \leq 0$ or $b \leq 0$.

(c) If $a > 0$, and $b > 0$ or $b < 0$, then $ab^2 > 0$.

If $ab^2 \leq 0$, then $a \leq 0$ or $b = 0$.

2. Prove the following theorem.

Theorem: Let a be an integer and let r be an irrational number.

Suppose $a \neq 0$ and $r \neq 0$.

Then ar is irrational.

Proof: Let a be an integer, and let r be an irrational number.

Suppose $a \neq 0$ and $r \neq 0$.

Suppose ar is rational.

Then there exist integers m and n with $n \neq 0$ such that $ar = \frac{m}{n}$.

Since a is non-zero integer, $\frac{1}{a}$ is a rational number, and so we have

$$\frac{1}{a}ar = \frac{1}{a} \frac{m}{n} = \frac{m}{na}$$

and, hence,

$$r = \frac{m}{na}.$$

Since n and a are integers, na is an integer.

Since n and a are non-zero, na is non-zero.

Hence, $\frac{m}{na}$ is a rational number, and so r is rational.

This is a contradiction to our supposition that r is irrational.

Hence, our supposition that ar is rational is false.

Thus, ar is irrational. ■

3. Prove the following theorem.

Theorem: Let a and b be positive integers.

If $a - 1$ is divisible by 4, and $b - 1$ is divisible by 4, then $ab - 1$ is divisible by 4.

Proof:

Let a and b be positive integers.

Suppose $4|a - 1$ and $4|b - 1$.

Then there exist integers k and m such that $a - 1 = 4k$ and $b - 1 = 4m$.

Then $a = 4k + 1$ and $b = 4m + 1$, so

$$ab - 1 = (4k + 1)(4m + 1) - 1 = 16km + 4k + 4m = 4(4km + k + m).$$

Since k and m are integers, $4km + k + m$ is an integer.

Hence, $4|ab - 1$. ■

4. Prove the following theorem.

Theorem: For all integers m , there exists a unique integer n such that $2m(m + 1) - 4n = 0$.

Proof:

Commentary: In order for $2m(m + 1) - 4n = 0$ to hold, we must have

$$n = \frac{m(m + 1)}{2}.$$

The trouble with this is that we need n to be an integer; hence, we must show that $m(m + 1)$ is even.

Let m be an integer.

We will show that $m(m + 1)$ is even.

Suppose m is even.

Then $m = 2k$ for some integer k .

Hence, $m(m + 1) = 2k(m + 1) = 2(k(m + 1))$.

Since $k(m + 1)$ is an integer, $m(m + 1)$ is even.

Now, suppose m is odd.

Then $m = 2k + 1$ for some integer k .

Hence, $m(m + 1) = m(2k + 1 + 1) = m(2k + 2) = 2(m(k + 1))$.

Since $m(k + 1)$ is an integer, $m(m + 1)$ is even.

Since $m(m + 1)$ is even, there exists an integer r such that $m(m + 1) = 2r$.

Let $n = r$.

Then $2m(m + 1) - 4n = 2m(m + 1) - 4r = 2m(m + 1) - 2(2r) = 2m(m + 1) - 2(m(m + 1)) = 0$.

Hence, there exists an integer n such that $2m(m + 1) - 4n = 0$.

To show uniqueness, suppose there is an integer n' such that $2m(m + 1) - 4n' = 0$.

Then

$$2m(m + 1) - 4n = 2m(m + 1) - 4n'$$

so

$$-4n = -4n'$$

so

$$-4(n - n') = 0.$$

Then $-4 = 0$ or $n - n' = 0$, but, since $-4 \neq 0$, we conclude $n - n' = 0$, i.e., $n = n'$.

Hence, the integer n for which $2m(m + 1) - 4n = 0$ is unique. ■

5. Prove the following theorem.

Theorem: Let $a \in \mathbb{Z}$. Then $|a| = 0$ iff $a = 0$.

Proof:

Let $a \in \mathbb{Z}$.

["Reverse" direction] Suppose $a = 0$.

Then $|a| = a = 0$.

(Thus, $a = 0$ implies $|a| = 0$.)

["Forward" direction, by contrapositive] Suppose $a \neq 0$.

Then $a > 0$ or $a < 0$.

Suppose $a > 0$.

Then $|a| = a > 0$, so $|a| > 0$, and so $|a| \neq 0$.

Suppose $a < 0$.

Then $|a| = -a$.

Since $a < 0$, $-a > 0$, and so $-a \neq 0$.

Hence, $|a| \neq 0$.

Thus, if $a \neq 0$, then $|a| \neq 0$.

(Therefore, $|a| = 0$ implies $a = 0$). ■

[Alternatively, for the "forward" direction, we can work directly and start by assuming $|a| = 0$.

Then from the definition of absolute value, we conclude that $a = 0$ or $-a = 0$.

If $-a = 0$, then $(-1)(-a) = (-1)(0) = 0$, so $a = 0$.

Hence, $a = 0$.]