

Math 300 B - Winter 2015

Midterm Exam Number One

January 28, 2015

Answers

1. Use a truth table to decide whether the following argument is valid. Explain your conclusion.

- Andie or Bing will sing.
- Bing and Clara will not both sing.
- Therefore, Andie and Bing will not both sing.

Let A be the statement “Andie will sing”, B be the statement “Bing will sing”, and C be the statement “Clara” will sing.

Then the above argument can be expressed as

- $A \vee B$
- $\neg(B \wedge C)$
- Therefore, $\neg(A \wedge B)$.

The conclusion is equivalent, by DeMorgan’s law, to $\neg A \vee \neg B$.

Then we have the following truth table for the argument.

A	B	C	$A \vee B$	$\neg B \vee \neg C$	$\neg A \vee \neg B$	
F	F	F	F	T	T	
F	F	T	F	T	T	
F	T	F	T	T	T	
F	T	T	T	F	T	
T	F	F	T	T	T	
T	F	T	T	T	T	
T	T	F	T	T	F	$\Leftarrow$
T	T	T	T	F	F	

The second-to-last row, marked with the arrow, shows that the premises can be true while the conclusion is false. Hence, the argument is invalid.

2. Let A, B , and C be sets. Verify the following identities by showing that the statement $x \in$ "the left hand set" is equivalent to the statement $x \in$ "the right hand set". Give a justification for each step (e.g., definition of union, DeMorgan's law, distributive law, etc.)

(a) $A \cap B = A \setminus (A \setminus B)$

$$\begin{aligned}
 & x \in A \setminus (A \setminus B) \\
 \Leftrightarrow & x \in A \wedge (x \notin A \setminus B) && \text{(def. of } \setminus \text{)} \\
 \Leftrightarrow & x \in A \wedge \neg(x \in A \setminus B) && \text{(def. of } \notin \text{)} \\
 \Leftrightarrow & x \in A \wedge \neg(x \in A \wedge x \notin B) && \text{(def. of } \setminus \text{)} \\
 \Leftrightarrow & x \in A \wedge (x \notin A \vee x \in B) && \text{(DeMorgan's law)} \\
 \Leftrightarrow & (x \in A \wedge x \notin A) \vee (x \in A \wedge x \in B) && \text{(distribution law)} \\
 \Leftrightarrow & x \in A \wedge x \in B && \text{(contradiction law)} \\
 \Leftrightarrow & x \in A \cap B && \text{(def. of } \cap \text{)}
 \end{aligned}$$

(b) $(A \setminus B) \cup (B \setminus C) \cup (B \cap C) = A \cup B$

$$\begin{aligned}
 & x \in (A \setminus B) \cup (B \setminus C) \cup (B \cap C) \\
 \Leftrightarrow & (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin C) \vee (x \in B \wedge x \in C) && \text{(def. of } \cup, \cap \text{ and } \setminus \text{)} \\
 \Leftrightarrow & (x \in A \wedge x \notin B) \vee (x \in B \wedge (x \notin C \vee x \in C)) && \text{(associative, distributive laws)} \\
 \Leftrightarrow & (x \in A \wedge x \notin B) \vee x \in B && \text{(tautology law)} \\
 \Leftrightarrow & (x \in A \vee x \in B) \wedge (x \notin B \vee x \in B) && \text{(distributive law)} \\
 \Leftrightarrow & x \in A \cup B && \text{(tautology law, def. of } \cup \text{)}
 \end{aligned}$$

3. Simplify the following expressions as much as possible. You should show a sequence of equivalent expressions connecting the original expression with your final one, and give a justification for each step (e.g., DeMorgan's law, distributive law, etc.)

(a) $P \wedge (Q \vee (P \wedge \neg Q))$

$$\begin{aligned} & P \wedge (Q \vee (P \wedge \neg Q)) \\ \Leftrightarrow & P \wedge ((Q \vee P) \wedge (Q \vee \neg Q)) && \text{(distributive law)} \\ \Leftrightarrow & P \wedge (Q \vee P) && \text{(tautology law)} \\ \Leftrightarrow & P && \text{(absorption law)} \end{aligned}$$

(b) $(P \vee Q) \wedge (\neg P \wedge (\neg Q \vee P))$

$$\begin{aligned} & (P \vee Q) \wedge (\neg P \wedge (\neg Q \vee P)) \\ \Leftrightarrow & (P \vee Q) \wedge ((\neg P \wedge \neg Q) \vee (\neg P \wedge P)) && \text{(distributive law)} \\ \Leftrightarrow & (P \vee Q) \wedge (\neg P \wedge \neg Q) && \text{(contradiction law)} \\ \Leftrightarrow & ((P \vee Q) \wedge \neg P) \wedge \neg Q && \text{(associative law)} \\ \Leftrightarrow & ((P \wedge \neg P) \vee (Q \wedge \neg P)) \wedge \neg Q && \text{(distributive law)} \\ \Leftrightarrow & (Q \wedge \neg P) \wedge \neg Q && \text{(contradiction law)} \\ \Leftrightarrow & (Q \wedge \neg Q) \wedge \neg P && \text{(commutative law, associative law)} \\ \Leftrightarrow & \text{contradiction} && \text{(contradiction law)} \end{aligned}$$

4. Simplify the following expressions as much as possible. You should show a sequence of equivalent expressions connecting the original expression with your final one, and give a justification for each step (e.g., DeMorgan's law, distributive law, etc.)

(a) $(\neg(P \wedge \neg Q) \rightarrow \neg P) \wedge Q$

$$\begin{aligned}
 & (\neg(P \wedge \neg Q) \rightarrow \neg P) \wedge Q \\
 \Leftrightarrow & ((P \wedge \neg Q) \vee \neg P) \wedge Q && \text{(conditional law)} \\
 \Leftrightarrow & ((P \vee \neg P) \wedge (\neg Q \vee \neg P)) \wedge Q && \text{(distributive law)} \\
 \Leftrightarrow & (\neg Q \vee \neg P) \wedge Q && \text{(tautology law)} \\
 \Leftrightarrow & (\neg Q \wedge Q) \vee (\neg P \wedge Q) && \text{(distributive law)} \\
 \Leftrightarrow & \neg P \wedge Q && \text{(contradiction law)}
 \end{aligned}$$

(b) $(Q \wedge P) \Leftrightarrow (Q \rightarrow P)$

$$\begin{aligned}
 & (Q \wedge P) \Leftrightarrow (Q \rightarrow P) \\
 \Leftrightarrow & (\neg(Q \wedge P) \vee (Q \rightarrow P)) \wedge (\neg(Q \rightarrow P) \vee (Q \wedge P)) && \text{(conditional law, def. of } \Leftrightarrow \text{)} \\
 \Leftrightarrow & (\neg Q \vee \neg P \vee (\neg Q \vee P)) \wedge ((Q \wedge \neg P) \vee (Q \wedge P)) && \text{(conditional law, DeMorgan's law)} \\
 \Leftrightarrow & (Q \wedge \neg P) \vee (Q \wedge P) && \text{(assoc., commut., taut. laws)} \\
 \Leftrightarrow & Q \wedge (\neg P \vee P) && \text{(distributive law)} \\
 \Leftrightarrow & Q && \text{(tautology law)}
 \end{aligned}$$

5. Write useful contrapositives of the following sentences. Express the contrapositives as sentences, not symbolically.

(a) If n is prime, then n is odd or $n = 2$.

If n is even and $n \neq 2$, then n is not prime.

(b) If I am hungry or cold, then I am sad and afraid.

If I am not sad or not afraid, then I am not hungry and I am not cold.

(c) If x is not zero, then there exists a number y such that $xy > 1$.

If for all y , $xy \leq 1$, then $x = 0$.