

Math 300 B - Winter 2015
Midterm Exam Number Two
February 25, 2015
Answers

1. Let A , B and C be sets. Prove that $(A \cup B) \cap C \subseteq A \cup (B \cap C)$.

Let A , B , and C be sets.

Suppose $x \in (A \cup B) \cap C$.

Then $x \in (A \cup B)$ and $x \in C$.

So, $x \in A$ or $x \in B$.

Suppose $x \in A$.

Then $x \in A \cup (B \cap C)$.

Suppose $x \in B$.

Then $x \in B \cap C$, so $x \in A \cup (B \cap C)$.

Therefore $x \in A \cup (B \cap C)$.

Thus, $x \in (A \cup B) \cap C$ implies $x \in A \cup (B \cap C)$.

Hence, $(A \cup B) \cap C \subseteq A \cup (B \cap C)$. ■

2. Let A and B be sets. Prove that $A \cap B = \emptyset$ if and only if $\mathcal{P}(A) \cap \mathcal{P}(B) = \{\emptyset\}$.

Let A and B be sets.

Suppose $A \cap B \neq \emptyset$.

Then there exists x such that $x \in A$ and $x \in B$.

Let $S = \{x\}$, the set containing x as its only element.

Since $x \in A$, $S \subseteq A$, so $S \in \mathcal{P}(A)$.

Since $x \in B$, $S \subseteq B$, so $S \in \mathcal{P}(B)$.

Thus, $S \in \mathcal{P}(A) \cap \mathcal{P}(B)$, and $S \neq \emptyset$, so $\mathcal{P}(A) \cap \mathcal{P}(B) \neq \{\emptyset\}$.

Suppose $\mathcal{P}(A) \cap \mathcal{P}(B) \neq \{\emptyset\}$.

Since $\emptyset \in \mathcal{P}(A) \cap \mathcal{P}(B)$, we can conclude that there is a set $S \in \mathcal{P}(A) \cap \mathcal{P}(B)$ and $S \neq \emptyset$.

Hence, there is an element $x \in S$.

Since $S \in \mathcal{P}(A)$, $S \subseteq A$, and so $x \in A$.

Since $S \in \mathcal{P}(B)$, $S \subseteq B$, and so $x \in B$.

Thus, $x \in A \cap B$, so $A \cap B \neq \emptyset$.

Hence, $A \cap B = \emptyset$ if and only if $\mathcal{P}(A) \cap \mathcal{P}(B) = \{\emptyset\}$. ■

3. Prove that, for all integers n , 3 does not divide $n^2 - 5$.

Let n be an integer.

By Euclid's theorem, $n = 3k + r$ for some integers k and r , and $0 \leq r < 3$.

That is, $r = 0, 1$, or 2 .

Then $n^2 - 5 = (3k + r)^2 - 5 = 9k^2 + 6kr + r^2 - 5 = 9k^2 + 6kr + r^2 - 6 + 1 = 3(3k^2 + 2kr - 2) + r^2 + 1$.

Let t be the remainder when $n^2 - 5$ is divided by 3.

Then t is equal to the remainder then $r^2 + 1$ is divided by 3.

If $r = 0$, then $r^2 + 1 = 1 = 0 \cdot 3 + 1$, so $t = 1$.

If $r = 1$, then $r^2 + 1 = 2 = 0 \cdot 3 + 2$, so $t = 2$.

If $r = 2$, then $r^2 + 1 = 5 = 1 \cdot 3 + 2$, so $t = 2$.

Hence the remainder when $n^2 - 5$ is divided by 3 is 1 or 2 and not zero.

Thus 3 does not divide $n^2 - 5$. ■

Here is an alternative proof using modular arithmetic.

Let n be an integer.

By Euclid's theorem, $n \equiv 0, 1$ or $2 \pmod{3}$.

Hence, $n^2 \equiv 0, 1$, or $4 \pmod{3}$; that is, $n^2 \equiv 0$ or $1 \pmod{3}$.

Also, $-5 \equiv 1 \pmod{3}$, since $-5 = (-2)(3) + 1$.

Hence, $n^2 - 5 \equiv 1$ or $2 \pmod{3}$.

Since $n^2 - 5$ cannot be congruent to zero modulo 3, $n^2 - 5$ is not divisible by 3. ■

4. Let A and B be sets. Prove that $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.

Let A and B be sets.

Suppose $S \in \mathcal{P}(A \cap B)$.

Then $S \subseteq A \cap B$.

Suppose $x \in S$.

Then $x \in A \cap B$, so $x \in A$ and $x \in B$.

Thus $x \in S$ implies $x \in A$, so $S \subseteq A$.

Also, $x \in S$ implies $x \in B$, so $S \subseteq B$.

Thus $S \in \mathcal{P}(A)$ and $S \in \mathcal{P}(B)$, so $S \in \mathcal{P}(A) \cap \mathcal{P}(B)$.

Thus $S \in \mathcal{P}(A \cap B)$ implies $S \in \mathcal{P}(A) \cap \mathcal{P}(B)$, so $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$.

Suppose $S \in \mathcal{P}(A) \cap \mathcal{P}(B)$.

Then $S \in \mathcal{P}(A)$ and $S \in \mathcal{P}(B)$.

So $S \subseteq A$ and $S \subseteq B$.

Suppose $x \in S$.

Then $x \in A$ and $x \in B$, so $x \in A \cap B$.

Thus, $x \in S$ implies $x \in A \cap B$, so $S \subseteq A \cap B$.

That is, $S \in \mathcal{P}(A \cap B)$.

Thus, $S \in \mathcal{P}(A) \cap \mathcal{P}(B)$ implies $S \in \mathcal{P}(A \cap B)$, so

$$\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B).$$

Since $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$ and $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$,

$$\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B). \blacksquare$$

5. Define a relation R on $\mathbb{Z}$ by

$$(x, y) \in R \Leftrightarrow 4 \mid x^2 - y^2.$$

Is R an equivalence relation? Prove your answer.

Let $x \in \mathbb{Z}$.

Then $x^2 - x^2 = 0 = 4 \cdot 0$, so $4 \mid x^2 - x^2$.

Hence, $(x, x) \in R$, and so R is reflexive.

Suppose $(x, y) \in R$.

Then $4 \mid x^2 - y^2$.

That is, $x^2 - y^2 = 4k$ for some integer k .

Then $y^2 - x^2 = -4k = 4(-k)$ and since $-k \in \mathbb{Z}$, $4 \mid y^2 - x^2$.

Hence, $(y, x) \in R$, and so R is symmetric.

Suppose $(x, y) \in R$ and $(y, z) \in R$.

Then $4 \mid x^2 - y^2$, so $x^2 - y^2 = 4k$ for some integer k .

Also, $4 \mid y^2 - z^2$, so $y^2 - z^2 = 4m$ for some integer m .

Hence, $x^2 - y^2 + y^2 - z^2 = x^2 - z^2 = 4k + 4m = 4(k + m)$.

Since $k + m$ is an integer, $4 \mid x^2 - z^2$, and so $(x, z) \in R$.

Thus, R is transitive.

Since R is reflexive, symmetric and transitive, R is an equivalence relation. ■