

Math 300 B - Spring 2025
Midterm Exam Number Two
May 21, 2025
Solutions

1. Let A and B be sets.

Prove that $A \cap B = A$ iff $A \cup B = B$.

Proof:

Suppose $A \cap B = A$.

Suppose $x \in A \cup B$.

Then $x \in A$ or $x \in B$.

Suppose $x \in A$.

Then $x \in A \cap B$, so $x \in B$.

Hence, $x \in A \cup B$ implies $x \in B$, so $A \cup B \subseteq B$.

Suppose $x \in B$.

Then $x \in A \cup B$.

Hence, $x \in B$ implies $x \in A \cup B$, so $B \subseteq A \cup B$.

Since $A \cup B \subseteq B$ and $B \subseteq A \cup B$, $A \cup B = B$.

So $A \cap B = A$ implies $A \cup B = B$.

Now, suppose $A \cup B = B$.

Suppose $x \in A \cap B$.

Then $x \in A$.

Hence, $x \in A \cap B$ implies $x \in A$, so $A \cap B \subseteq A$.

Suppose $x \in A$.

Then $x \in A \cup B$, so $x \in B$, so $x \in A \cap B$.

Hence, $x \in A$ implies $x \in A \cap B$, so $A \subseteq A \cap B$.

Since $A \cap B \subseteq A$ and $A \subseteq A \cap B$, $A \cap B = A$.

So, $A \cup B = B$ implies $A \cap B = A$. ■

2. Prove the following.

Suppose A and B are sets. Suppose $A \not\subseteq B$ and $B \not\subseteq A$. Then

$$\mathcal{P}(A) \cup \mathcal{P}(B) \neq \mathcal{P}(A \cup B).$$

Proof:

Suppose A and B are sets with $A \not\subseteq B$ and $B \not\subseteq A$.

Then $\exists x \in A$ such that $x \notin B$, and $\exists y \in B$ such that $y \notin A$.

Let $S = \{x, y\}$.

Since $a \in S$ implies $a \in A \cup B$, $S \subseteq A \cup B$.

So, $S \in \mathcal{P}(A \cup B)$.

On the other hand, since $y \notin A$, $S \not\subseteq A$.

Also, since $x \notin B$, $S \not\subseteq B$.

So $S \notin \mathcal{P}(A)$ and $S \notin \mathcal{P}(B)$.

That is, $S \notin \mathcal{P}(A) \cup \mathcal{P}(B)$.

Since $S \in \mathcal{P}(A \cup B)$ but $S \notin \mathcal{P}(A) \cup \mathcal{P}(B)$, we conclude that $\mathcal{P}(A) \cup \mathcal{P}(B) \neq \mathcal{P}(A \cup B)$. ■

3. Let $\mathcal{F}$ and $\mathcal{G}$ be families. Prove that

$$\bigcup(\mathcal{F} \cap \mathcal{G}) \subseteq \left(\bigcup \mathcal{F}\right) \cap \left(\bigcup \mathcal{G}\right).$$

Proof:

Let $\mathcal{F}$ and $\mathcal{G}$ be families.

Suppose $x \in \bigcup(\mathcal{F} \cap \mathcal{G})$.

Then $\exists S \in \mathcal{F} \cap \mathcal{G}$ such that $x \in S$.

So $S \in \mathcal{F}$ and $S \in \mathcal{G}$.

So $x \in \bigcup \mathcal{F}$ and $x \in \bigcup \mathcal{G}$.

So $x \in (\bigcup \mathcal{F}) \cap (\bigcup \mathcal{G})$.

Thus, $x \in \bigcup(\mathcal{F} \cap \mathcal{G})$ implies $x \in (\bigcup \mathcal{F}) \cap (\bigcup \mathcal{G})$.

Hence, $\bigcup(\mathcal{F} \cap \mathcal{G}) \subseteq (\bigcup \mathcal{F}) \cap (\bigcup \mathcal{G})$. ■

4. Use induction to prove that $2^n \geq n^2$ for all integers $n \geq 4$.

Proof:

Let $P(n) = "2^n \geq n^2"$ for integers $n \geq 4$.

Base case: Let $n = 4$.

Then $2^n = 2^4 = 16 = 4^2 = n^2$.

So $2^n \geq n^2$, so $P(4)$ is true.

Induction step: Suppose $P(x)$ is true for some integer $x \geq 4$.

So $2^x \geq x^2$.

Then:

$$\begin{aligned} 2^{x+1} &= 2 \cdot 2^x \geq 2x^2 \\ &= x^2 + x^2 \\ &\geq x^2 + 4x \quad (\text{since } x \geq 4) \\ &= x^2 + 2x + 2x \\ &\geq x^2 + 2x + 8 \quad (\text{since } x \geq 4) \\ &> x^2 + 2x + 1 \\ &= (x+1)^2. \end{aligned}$$

So $P(x+1)$ is true.

Hence, $P(x)$ implies $P(x+1)$, and, since $P(4)$ is true, $P(n)$ is true for all integers $n \geq 4$ by induction. ■

5. Define a relation R on the integers $\mathbb{Z}$ by

$$(a, b) \in R \text{ iff } ab \geq 0.$$

Is R reflexive?

Is R symmetric?

Is R transitive?

Is R an equivalence relation?

Prove your answers are correct.

Reflexivity Let $a \in \mathbb{Z}$.

Then $a^2 \geq 0$, so $(a, a) \in R$.

Hence, for all $a \in \mathbb{Z}$, $(a, a) \in R$, so R is reflexive.

Symmetry Suppose $(a, b) \in R$.

Then $ba = ab \geq 0$, so $(b, a) \in R$.

Hence, $(a, b) \in R$ implies $(b, a) \in R$, so R is symmetric.

Transitivity We note that $(1, 0) \in R$ and $(0, -1) \in R$, but $(1, -1) \notin R$ since $(1)(-1) \not\geq 0$.

Hence, R is not transitive.

Since R is not transitive, R is not an equivalence relation.

Elementary Properties of the Integers (EPIs)

Suppose a, b, c , and d are integers.

- **Closure:** $a + b$ and ab are integers.
 - **Substitution of Equals:** If $a = b$, then $a + c = b + c$ and $ac = bc$.
 - **Commutativity:** $a + b = b + a$ and $ab = ba$.
 - **Associativity:** $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$.
 - **The Distributive Law:** $a(b + c) = ab + ac$
 - **Identities:** $a + 0 = 0 + a = a$ and $a \cdot 1 = 1 \cdot a = a$.
 0 is called the additive identity.
 1 is called the multiplicative identity.
 - **Additive Inverses:** There exists an integer $-a$ such that $a + (-a) = (-a) + a = 0$.
 - **Trichotomy:** Exactly one of the following is true:
 $a > 0$, $-a > 0$, or $a = 0$.
 - **The Well-Ordering Principle:** Every non-empty set of positive integers contains a smallest element.
1. $a \cdot 0 = 0$
 2. If $a + c = b + c$, then $a = b$.
 3. $-a = (-1) \cdot a$
 4. $(-a) \cdot b = -(ab)$
 5. $(-a) \cdot (-b) = ab$
 6. If $ab = 0$, then $a = 0$ or $b = 0$.
 7. If $a \leq b$ and $b \leq a$, then $a = b$.
 8. If $a < b$ and $b < c$, then $a < c$.
 9. If $a < b$, then $a + c < b + c$.
 10. If $a < b$ and $0 < c$, then $ac < bc$.
 11. If $a < b$ and $c < 0$, then $bc < ac$.
 12. If $a < b$ and $c < d$, then $a + c < b + d$.
 13. If $0 \leq a < b$ and $0 \leq c < d$, then $ac < bd$.
 14. If $a < b$, then $-b < -a$.
 15. $0 \leq a^2$, where $a^2 = a \cdot a$.
 16. If $ab = 1$, then either $a = b = 1$ or $a = b = -1$.

Sets

$A \subseteq B$ iff $x \in A$ implies $x \in B$

$A = B$ iff $A \subseteq B$ and $B \subseteq A$

$x \in A \cup B$ iff $x \in A$ or $x \in B$

$x \in A \cap B$ iff $x \in A$ and $x \in B$

$x \in A \setminus B$ iff $x \in A$ and $x \notin B$

$\mathcal{P}(A)$ is the set of all subsets of a set A

NOTE: Properties 8-14 hold if each $<$ is replaced with $\leq$.