

Math 300 B - Spring 2025
Midterm Exam Number One
April 23, 2025
Solutions

1. *Using the Elementary Properties of the Integers, carefully prove the following theorem.*

Theorem: Let $a, b \in \mathbb{Z}$. Then $(a + b)^2 = a^2 + (2ab + b^2)$.

Proof: Let $a, b \in \mathbb{Z}$.

Then

$$\begin{aligned}(a + b)^2 &= (a + b)(a + b) && \text{(notation)} \\&= (a + b)a + (a + b)b && \text{(by the Distributive Law)} \\&= a(a + b) + b(a + b) && \text{(by Commutativity and Substitution of Equals)} \\&= (a^2 + ab) + (ba + b^2) && \text{(by the Distributive Law and Substitution of Equals)} \\&= (a^2 + ab) + (ab + b^2) && \text{(by Commutativity and Substitution of Equals)} \\&= ((a^2 + ab) + ab) + b^2 && \text{(by Associativity)} \\&= ((a^2 + (ab + ab)) + b^2) && \text{(by Associativity and Substitution of Equals)} \\&= (a^2 + 2ab) + b^2 && \text{(notation and Substitution of Equals)} \\&= a^2 + (2ab + b^2). && \text{(by Associativity)}\end{aligned}$$

Thus, by Transitivity of Equals, $(a + b)^2 = a^2 + (2ab + b^2)$.

2. Prove the following theorem.

Theorem: Let $x, y \in \mathbb{Z}$.

Then $|x - y| > x - y$ if and only if $x < y$.

(That is, you need to prove two things:

(a) if $|x - y| > x - y$ then $x < y$, and

(b) if $x < y$ then $|x - y| > x - y$.)

Proof:

Let $x, y \in \mathbb{Z}$.

Suppose $|x - y| > x - y$.

Suppose $x - y \geq 0$.

Then $|x - y| = x - y \not> x - y$.

This contradicts our assumption that $|x - y| > x - y$

Hence, $x - y < 0$, and so $x < y$.

Now, suppose $x < y$.

Then $x - y < 0$, so $|x - y| = -(x - y) > 0 > x - y$.

So, $|x - y| > x - y$. ■

3. Prove the following theorem.

Theorem: Let a be a rational number and let b be an irrational number.

Then $3a + b$ is an irrational number.

Proof:

Let a be a rational number and let b be an irrational number.

Suppose $3a + b \in \mathbb{Q}$.

Then $3a + b = \frac{r}{s}$ for some $r, s \in \mathbb{Z}$ with $s \neq 0$.

Then $b = \frac{r}{s} - 3a$.

Since a is rational, $a = \frac{m}{n}$ for some $m, n \in \mathbb{Z}$ with $n \neq 0$.

Then $b = \frac{r}{s} - 3\left(\frac{m}{n}\right) = \frac{rn - 3ms}{ns}$.

We have $rn - 3ms \in \mathbb{Z}$ and $ns \in \mathbb{Z}$ by closure of integer addition and multiplication.

Also, $ns \neq 0$ since $s \neq 0$ and $n \neq 0$.

Hence, b is a rational number, which contradicts our assumption that b is irrational.

Thus, the assumption that $3a + b \in \mathbb{Q}$ is false.

Hence, $3a + b$ is irrational. ■

Alternate proof: We may also prove this using the fact that the rationals are closed under addition and multiplication, and each rational has an additive inverse.

Suppose a is rational and b is irrational.

Suppose $3a + b$ is rational.

Then $3a$ is rational, and $-3a$ is rational, so $(3a + b) + (-3a)$ is rational.

That is, b is rational.

This contradicts our assumption that b is irrational, and so we conclude that $3a + b$ is irrational. ■

4. Prove the following theorem.

Theorem: Let a and b be positive integers.

If $ab \mid (a + b)$ then $a = b$.

Proof:

Let a and b be positive integers.

Suppose $ab \mid (a + b)$.

Then $a + b = abk$ for some $k \in \mathbb{Z}$.

Then $a = abk - b = b(ak - 1)$ and $b = abk - a = a(bk - 1)$.

Since $ak - 1$ and $bk - 1$ are integers, we conclude that $a \mid b$ and $b \mid a$.

For simplicity, let $r = ak - 1$ and $s = bk - 1$.

So $b = ar$ and $a = bs$, and hence $a = ars$.

Then $a(1 - rs) = 0$.

Since $a > 0$, we have $1 - rs = 0$ and so $rs = 1$, and hence $r = s = 1$ or $r = s = -1$.

If $r = -1$, then $b = ar < 0$. This is a contradiction.

Hence $r = s = 1$ and thus $a = b$. ■