

Math 300 C - Spring 2022
Midterm Exam Number Two
May 18, 2022
Solutions

1. Prove the following theorem.

Theorem: Let A , B , and C be sets. Suppose $A \cup C \subseteq B \cup C$. Then $A \setminus C \subseteq B$.

Proof: Let A , B , and C be sets.

Suppose $A \cup C \subseteq B \cup C$.

Suppose $x \in A \setminus C$.

Then $x \in A$ and $x \notin C$.

Since $x \in A$, $x \in A \cup C$, and so $x \in B \cup C$ (since $A \cup C \subseteq B \cup C$).

So, $x \in B$ or $x \in C$.

But $x \notin C$, so $x \in B$.

Thus, $x \in A \setminus C$ implies $x \in B$, and hence $A \setminus C \subseteq B$. ■

2. Prove the following theorem.

Theorem: Suppose $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are families of sets, with $\mathcal{A} \neq \emptyset$, $\mathcal{B} \neq \emptyset$, and $\mathcal{C} \neq \emptyset$. Suppose that for every $S \in \mathcal{A}$ and $T \in \mathcal{B}$, $S \cup T \in \mathcal{C}$.

Then $\bigcap \mathcal{C} \subseteq (\bigcap \mathcal{A}) \cup (\bigcap \mathcal{B})$.

Proof 1: Suppose $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are families of sets, with $\mathcal{A} \neq \emptyset$, $\mathcal{B} \neq \emptyset$, and $\mathcal{C} \neq \emptyset$. Suppose that for every $S \in \mathcal{A}$ and $T \in \mathcal{B}$, $S \cup T \in \mathcal{C}$.

Suppose $x \in \bigcap \mathcal{C}$.

Suppose $x \notin \bigcap \mathcal{A}$.

Then $\exists S \in \mathcal{A}$ such that $x \notin S$.

Let $T \in \mathcal{B}$.

Then $S \cup T \in \mathcal{C}$, so $x \in S \cup T$ since $x \in \bigcap \mathcal{C}$.

Since $x \in S \cup T$ and $x \notin S$, $x \in T$.

Thus, for all $T \in \mathcal{B}$, $x \in T$, so $x \in \bigcap \mathcal{B}$.

Thus, $x \notin \bigcap \mathcal{A}$ implies $x \in \bigcap \mathcal{B}$.

Thus $x \in \bigcap \mathcal{A}$ or $x \in \bigcap \mathcal{B}$, so $x \in (\bigcap \mathcal{A}) \cup (\bigcap \mathcal{B})$.

thus, $x \in \bigcap \mathcal{C}$ implies $x \in (\bigcap \mathcal{A}) \cup (\bigcap \mathcal{B})$, so $\bigcap \mathcal{C} \subseteq (\bigcap \mathcal{A}) \cup (\bigcap \mathcal{B})$. ■

Proof 2: Suppose $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are families of sets, with $\mathcal{A} \neq \emptyset$, $\mathcal{B} \neq \emptyset$, and $\mathcal{C} \neq \emptyset$. Suppose that for every $S \in \mathcal{A}$ and $T \in \mathcal{B}$, $S \cup T \in \mathcal{C}$.

Suppose $x \in \bigcap \mathcal{C}$.

Suppose $x \notin (\bigcap \mathcal{A}) \cup (\bigcap \mathcal{B})$, so $x \notin \bigcap \mathcal{A}$ and $x \notin \bigcap \mathcal{B}$.

Then $\exists S \in \mathcal{A}$ such that $x \notin S$.

Also, $\exists T \in \mathcal{B}$ such that $x \notin T$.

Hence, $x \notin S \cup T$.

But, $S \cup T \in \mathcal{C}$, and $x \in \bigcap \mathcal{C}$, so $x \in S \cup T$.

Thus, we have a contradiction.

Hence, supposing $x \notin (\bigcap \mathcal{A}) \cup (\bigcap \mathcal{B})$ led us to a contradiction, so this statement is false, and hence $x \in (\bigcap \mathcal{A}) \cup (\bigcap \mathcal{B})$.

Thus, $x \in \bigcap \mathcal{C}$ implies $x \in (\bigcap \mathcal{A}) \cup (\bigcap \mathcal{B})$, so $\bigcap \mathcal{C} \subseteq (\bigcap \mathcal{A}) \cup (\bigcap \mathcal{B})$. ■

Proof 3: Suppose $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are families of sets, with $\mathcal{A} \neq \emptyset$, $\mathcal{B} \neq \emptyset$, and $\mathcal{C} \neq \emptyset$.

Suppose $\bigcap \mathcal{C} \not\subseteq (\bigcap \mathcal{A}) \cup (\bigcap \mathcal{B})$.

Then $\exists x \in \bigcap \mathcal{C}$ such that $x \notin \bigcap \mathcal{A}$ and $x \notin \bigcap \mathcal{B}$.

Then $\exists S \in \mathcal{A}$ such that $x \notin S$, and $\exists T \in \mathcal{B}$ such that $x \notin T$.

Then $x \notin S \cup T$.

Since $x \notin S \cup T$, $S \cup T \notin \mathcal{C}$.

(That is, there exist $S \in \mathcal{A}$ and $T \in \mathcal{B}$ such that $S \cup T \notin \mathcal{C}$.) ■

3. Use induction to prove the following theorem.

Theorem: For all integers $n \geq 1$,

$$\sum_{j=1}^n \frac{1}{(2j-1)(2j+1)} = \frac{n}{2n+1}.$$

Proof: Let $P(n) = “ \sum_{j=1}^n \frac{1}{(2j-1)(2j+1)} = \frac{n}{2n+1} ”$.

Base case: Let $n = 1$.

Then

$$\begin{aligned} \sum_{j=1}^n \frac{1}{(2j-1)(2j+1)} &= \sum_{j=1}^1 \frac{1}{(2j-1)(2j+1)} \\ &= \frac{1}{2(1)-1)(2(1)+1)} = \frac{1}{1 \cdot 3} = \frac{1}{3} = \frac{1}{2 \cdot 1 + 1} \\ &= \frac{n}{2n+1}. \end{aligned}$$

So $P(1)$ is true.

Induction step: Suppose $P(n)$ is true for some $n = k \geq 1$. Then

$$\begin{aligned} \sum_{j=1}^{k+1} \frac{1}{(2j-1)(2j+1)} &= \sum_{j=1}^k \frac{1}{(2j-1)(2j+1)} + \frac{1}{(2(k+1)-1)(2(k+1)+1)} \\ &= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} \\ &= \frac{k(2k+3)+1}{(2k+1)(2k+3)} = \frac{2k^2+3k+1}{(2k+1)(2k+3)} \\ &= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)} = \frac{k+1}{2k+3} = \frac{k+1}{2(k+1)+1}. \end{aligned}$$

Thus, $P(k+1)$ is true.

Hence, $P(n)$ implies $P(n+1)$ for all $n \geq 1$, and, since $P(1)$ is true, $P(n)$ is true for all $n \geq 1$ by induction. ■

4. Define the relation $R \subseteq \mathbb{Z} \times \mathbb{Z}$ by

$$(x, y) \in R \text{ if and only if } 3 \mid (x - y).$$

Is R reflexive?

Is R symmetric?

Is R transitive?

Is R an equivalence relation?

(Note you have **four** questions to answer here.)

Reflexivity: Suppose $x \in \mathbb{Z}$.

Then $x - x = 0 = 3 \cdot 0$, so $3 \mid x - x$.

Hence, $(x, x) \in R$.

Thus, $(x, x) \in R$ for all $x \in \mathbb{Z}$, so R is reflexive.

Symmetry: Suppose $(x, y) \in R$.

Then $3 \mid x - y$, so $x - y = 3k$ for some integer k .

Hence, $y - x = -(x - y) = -3k = 3(-k)$, so $3 \mid y - x$.

Thus, $(y, x) \in R$.

So $(x, y) \in R$ implies $(y, x) \in R$ and hence R is symmetric.

Transitivity: Suppose $(x, y) \in R$ and $(y, z) \in R$.

Then $3 \mid x - y$ and $3 \mid y - z$ so $x - y = 3k$ and $y - z = 3m$ for some integers k and m .

Then $x - z = x - y + (y - z) = 3k + 3m = 3(k + m)$.

Since $k + m$ is an integer, $3 \mid x - z$, and hence $(x, z) \in R$.

Thus, when $(x, y) \in R$ and $(y, z) \in R$, $(x, z) \in R$, so R is transitive.

Since R is reflexive, symmetric and transitive, R is an equivalence relation.