

Math 300 C - Spring 2022
Midterm Exam Number One
April 20, 2022
Solutions

Name: _____

Student ID no. : _____

Signature: _____

1	10	
2	10	
3	10	
4	10	
Total	40	

- You have 50 minutes to complete the exam.

1. Prove the following theorem.

Theorem: Let

$$A = \{n \in \mathbb{Z} : 5n + 6 \geq 11\}$$

$$B = \{n \in \mathbb{Z} : 3n - 9 \leq 0\},$$

and

$$C = \{n \in \mathbb{Z} : n^2 - n \leq 6\}.$$

Then $A \cap B \subseteq C$.

Proof:

Let A , B and C be defined as above.

Suppose $x \in A \cap B$.

Then $x \in A$, so $5x + 6 \geq 11$, which implies that $5(x - 1) \geq 0$.

Since $5 > 0$, we conclude $x - 1 \geq 0$, so $x \geq 1$.

Also, $x \in B$, so $3x - 9 \leq 0$, i.e., $3(x - 3) \leq 0$.

Since $3 > 0$, we conclude $x - 3 \leq 0$, so $x \leq 3$.

Thus $1 \leq x \leq 3$, and since $x \in \mathbb{Z}$, this means $x = 1, 2$, or 3 .

If $x = 1$, then $x^2 - x = 0 \leq 6$, so $x \in C$.

If $x = 2$, then $x^2 - x = 2 \leq 6$, so $x \in C$.

If $x = 3$, then $x^2 - x = 6 \leq 6$, so $x \in C$.

Thus, $x \in C$.

So, we have shown that $x \in A \cap B$ implies $x \in C$.

Therefore, $A \cap B \subseteq C$. ■

Commentary:

Note that here we do not have to work with C in any way except to check that 1, 2 and 3 are in it.

We *could* solve the inequality $n^2 - n \leq 6$ to find that this is equivalent to $-2 \leq n \leq 3$, and we can see that 1, 2 and 3 satisfy this inequality. However, this is not necessary at all, and solving a quadratic inequality leads some people into trouble.

2. Prove the following theorem.

Theorem: Let a and b be integers. If $a|b| = |a|b$ then $ab \geq 0$.

Proof: Let a and b be integers.

We will prove the contrapositive of the statement above: if $ab < 0$ then $a|b| \neq |a|b$.

Suppose $ab < 0$.

If a and $b \geq 0$, or if $a < 0$ and $b < 0$, then we would have $ab \geq 0$.

Hence, we have two cases to consider:

(a) $a > 0$ and $b < 0$

(b) $a < 0$ and $b > 0$

Suppose $a > 0$ and $b < 0$.

Then $a|b| = -ab$ and $|a|b = ab$.

Suppose $-ab = ab$.

Then $2ab = 0$, so $a = 0$ or $b = 0$.

This is a contradiction to $a > 0$ and $b < 0$, so $-ab \neq ab$, i.e., $a|b| \neq |a|b$.

Suppose $a < 0$ and $b > 0$.

Then $a|b| = ab$ and $|a|b = -ab$.

Suppose $-ab = ab$.

Then, as above, we have a contradiction.

So, $-ab \neq ab$, i.e., $a|b| \neq |a|b$.

Thus, $a|b| \neq |a|b$. ■

3. Prove the following theorem.

Theorem: Let n be an integer. Then $10|n$ if and only if $2|n$ and $5|n$.

Proof: Let n be an integer.

Suppose $10|n$.

Then $n = 10k$ for some integer k .

Hence, $n = 2(5k)$, so $2|n$.

Also, $n = 5(2k)$, so $5|n$.

Thus, $10|n$ implies $2|n$ and $5|n$.

Suppose $2|n$ and $5|n$.

Then $n = 2k = 5m$ for some integers k and m .

So, n is even.

Suppose m is odd.

Then $m = 2r + 1$ for some integer r .

Then $n = 5(2r + 1) = 10r + 5 = 10r + 4 + 1 = 2(5r + 2) + 1$, so n is odd.

This is a contradiction to the fact that n is even.

Hence, m is even, so we can write m as $m = 2s$ for some integer s .

Then $n = 5m = 5(2s) = 10s$, and so $10|n$.

Thus, if $2|n$ and $5|n$, then $10|n$.

Therefore, $10|n$ if and only if $2|n$ and $5|n$. ■

Commentary:

The key step here is proving that since $5m$ is even, m must be even.

We might also argue that, since $2k$ is divisible by 5, then k must be divisible by 5. This, however, is quite a bit harder, since we don't have a good way to describe "is not divisible by 5". Whereas, we do have a good way to describe "is not divisible by 2".

We will develop more proof machinery in the next weeks to make the 5 argument more doable.

4. Prove the following theorem.

Theorem: For all pairs of even integers a and b , there exists an integer c such that

$$|a - c| = |b - c|.$$

(For this proof, you may use the fact that $|xy| = |x||y|$ for any integers x and y without proving it.)

Proof:

Let a and b be even integers.

Then $a = 2k$ and $b = 2m$ for some integers k and m .

Let $c = k + m$.

Then $|a - c| = |2k - (k + m)| = |k - m|$ and

$$\begin{aligned} |b - c| &= |2m - (k + m)| = |m - k| \\ &= |(-1)(k - m)| \\ &= |-1||k - m| \\ &= (1)|k - m| \\ &= |k - m|. \end{aligned}$$

Thus, $|a - c| = |b - c|$.

Hence, since $k + m \in \mathbb{Z}$, there exists an integer c such that $|a - c| = |b - c|$. ■

Commentary:

Why let $c = k + m$? The equation

$$|a - c| = |b - c|$$

says that the distance from a to c is the distance from b to c on the real line. Hence, c must be directly in the middle between a and b . In other words, c must be the average of a and b . With $a = 2k$ and $b = 2m$, the average of a and b is

$$\frac{a + b}{2} = \frac{2k + 2m}{2} = k + m.$$

Note that we cannot have c greater than both a and b or less than both a and b . If $c < a < b$, for example, then $|b - c| = b - c > a - c = |a - c|$. So c must be between a and b .

If we suppose that $a < c < b$, then $|a - c| = |b - c|$ implies

$$c - a = b - c$$

which implies $c = \frac{a + b}{2}$, so $c = k + m$ is a good thing to try.

Elementary Properties of the Integers (EPIs)

Suppose a, b, c , and d are integers.

1. **Closure:** $a + b$ and ab are integers.
2. **Substitution of Equals:** If $a = b$, then $a + c = b + c$ and $ac = bc$.
3. **Commutativity:** $a + b = b + a$ and $ab = ba$.
4. **Associativity:** $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$.
5. **The Distributive Law:** $a(b + c) = ab + ac$
6. **Identities:** $a + 0 = 0 + a = a$ and $a \cdot 1 = 1 \cdot a = a$.
0 is called the additive identity.
1 is called the multiplicative identity.
7. **Additive Inverses:** There exists an integer $-a$ such that $a + (-a) = (-a) + a = 0$.
8. **Trichotomy:** Exactly one of the following is true:
 $a > 0$, $-a > 0$, or $a = 0$.
9. **The Well-Ordering Principle:** Every non-empty set of positive integers contains a smallest element.
10. $a \cdot 0 = 0$
11. If $a + c = b + c$, then $a = b$.
12. $-a = (-1) \cdot a$
13. $(-a) \cdot b = -(ab)$
14. $(-a) \cdot (-b) = ab$
15. If $ab = 0$, then $a = 0$ or $b = 0$.
16. If $a \leq b$ and $b \leq a$, then $a = b$.
17. If $a < b$ and $b < c$, then $a < c$.
18. If $a < b$, then $a + c < b + c$.
19. If $a < b$ and $0 < c$, then $ac < bc$.
20. If $a < b$ and $c < 0$, then $bc < ac$.
21. If $a < b$ and $c < d$, then $a + c < b + d$.
22. If $0 \leq a < b$ and $0 \leq c < d$, then $ac < bd$.
23. If $a < b$, then $-b < -a$.
24. $0 \leq a^2$, where $a^2 = a \cdot a$.
25. If $ab = 1$, then either $a = b = 1$ or $a = b = -1$.

Sets

- $A \subseteq B$ iff $x \in A$ implies $x \in B$
 $A = B$ iff $A \subseteq B$ and $B \subseteq A$
 $x \in A \cup B$ iff $x \in A$ or $x \in B$
 $x \in A \cap B$ iff $x \in A$ and $x \in B$
 $x \in A \setminus B$ iff $x \in A$ and $x \notin B$

NOTE: Properties 17-23 hold if each $<$ is replaced with $\leq$.

One theorem for reference:

Theorem DAS (Divisors are Smaller):

Let a and b be positive integers. Then $a|b$ implies $a \leq b$.