

Math 300 C - Spring 2022
Midterm Exam Number One
April 20, 2022

Name: _____

Student ID no. : _____

Signature: _____

1	10	
2	10	
3	10	
4	10	
Total	40	

- You have 50 minutes to complete the exam.

1. Prove the following theorem.

Theorem: Let

$$A = \{n \in \mathbb{Z} : 5n + 6 \geq 11\}$$

$$B = \{n \in \mathbb{Z} : 3n - 9 \leq 0\},$$

and

$$C = \{n \in \mathbb{Z} : n^2 - n \leq 6\}.$$

Then $A \cap B \subseteq C$.

2. Prove the following theorem.

Theorem: Let a and b be integers. If $a|b| = |a|b$ then $ab \geq 0$.

3. Prove the following theorem.

Theorem: Let n be an integer. Then $10|n$ if and only if $2|n$ and $5|n$.

4. Prove the following theorem.

Theorem: For all pairs of even integers a and b , there exists an integer c such that

$$|a - c| = |b - c|.$$

(For this proof, you may use the fact that $|xy| = |x||y|$ for any integers x and y without proving it.)

Elementary Properties of the Integers (EPIs)

Suppose a, b, c , and d are integers.

1. **Closure:** $a + b$ and ab are integers.
2. **Substitution of Equals:** If $a = b$, then $a + c = b + c$ and $ac = bc$.
3. **Commutativity:** $a + b = b + a$ and $ab = ba$.
4. **Associativity:** $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$.
5. **The Distributive Law:** $a(b + c) = ab + ac$
6. **Identities:** $a + 0 = 0 + a = a$ and $a \cdot 1 = 1 \cdot a = a$.
0 is called the additive identity.
1 is called the multiplicative identity.
7. **Additive Inverses:** There exists an integer $-a$ such that $a + (-a) = (-a) + a = 0$.
8. **Trichotomy:** Exactly one of the following is true:
 $a > 0$, $-a > 0$, or $a = 0$.
9. **The Well-Ordering Principle:** Every non-empty set of positive integers contains a smallest element.
10. $a \cdot 0 = 0$
11. If $a + c = b + c$, then $a = b$.
12. $-a = (-1) \cdot a$
13. $(-a) \cdot b = -(ab)$
14. $(-a) \cdot (-b) = ab$
15. If $ab = 0$, then $a = 0$ or $b = 0$.
16. If $a \leq b$ and $b \leq a$, then $a = b$.
17. If $a < b$ and $b < c$, then $a < c$.
18. If $a < b$, then $a + c < b + c$.
19. If $a < b$ and $0 < c$, then $ac < bc$.
20. If $a < b$ and $c < 0$, then $bc < ac$.
21. If $a < b$ and $c < d$, then $a + c < b + d$.
22. If $0 \leq a < b$ and $0 \leq c < d$, then $ac < bd$.
23. If $a < b$, then $-b < -a$.
24. $0 \leq a^2$, where $a^2 = a \cdot a$.
25. If $ab = 1$, then either $a = b = 1$ or $a = b = -1$.

Sets

- $A \subseteq B$ iff $x \in A$ implies $x \in B$
 $A = B$ iff $A \subseteq B$ and $B \subseteq A$
 $x \in A \cup B$ iff $x \in A$ or $x \in B$
 $x \in A \cap B$ iff $x \in A$ and $x \in B$
 $x \in A \setminus B$ iff $x \in A$ and $x \notin B$

NOTE: Properties 17-23 hold if each $<$ is replaced with $\leq$.

One theorem for reference:

Theorem DAS (Divisors are Smaller):

Let a and b be positive integers. Then $a|b$ implies $a \leq b$.