

Math 300 C - Spring 2016
Midterm Exam Number Two
May 18, 2016
Solutions

1. Let $A = \mathbb{R} \times \mathbb{R} \setminus \{(0, 0)\}$.

Thus, A is the xy -plane without the origin.

Define a relation R on A by

$((x_1, y_1), (x_2, y_2)) \in R \Leftrightarrow (x_1, y_1) \text{ and } (x_2, y_2) \text{ lie on a line which passes through the origin.}$

Prove that R is transitive.

Suppose $P = (x_1, y_1) \in A$ and $Q = (x_2, y_2) \in A$ and $S = (x_3, y_3) \in A$ and $(P, Q) \in R$ and $(Q, S) \in R$.

Suppose $x_1 = 0$.

Then P lies on the vertical line $x = 0$ through the origin and no other line through the origin.

Hence, Q lies on $x = 0$, and hence S lies on $x = 0$.

Thus, P and S lie on a line through the origin, and so $(P, S) \in R$.

Suppose $x_1 \neq 0$.

Then P and Q lie on a line $y = mx$ where $m \in \mathbb{R}$, and Q and S lie on a line $y = nx$ where $n \in \mathbb{R}$.

Then $y_2 = mx_2 = nx_2$ so $(m - n)x_2 = 0$.

If $x_2 = 0$, then $y_2 = mx_2 = 0$, so $Q = (0, 0) \notin A$, a contradiction since $Q \in A$.

Hence $x_2 \neq 0$, so $m - n = 0$, i.e., $m = n$.

Thus, P and Q lie on the line $y = mx$ and Q and S lie on the line $y = mx$, so P and S both lie on a line through the origin.

Hence, $(P, S) \in R$ and thus R is transitive.

2. Let A , B , and C be sets.

Let $f : A \rightarrow C$ and $g : B \rightarrow C$.

Prove that $f \cup g : A \cup B \rightarrow C$ iff $f(x) = g(x)$ for all $x \in A \cap B$.

Suppose $f \cup g : A \cup B \rightarrow C$.

Suppose $\exists a \in A \cap B$ such that $f(a) \neq g(a)$.

Then, since $a \in A$, $(a, f(a)) \in f$ and, since $a \in B$, $(a, g(a)) \in g$.

Hence, $(a, f(a)) \in f \cup g$ and $(a, g(a)) \in f \cup g$.

But $f(a) \neq g(a)$ and hence $\nexists! y \in C$ such that $(a, y) \in f \cup g$, so $f \cup g$ is not a function.

This is a contradiction to our assumption that $f \cup g : A \cup B \rightarrow C$.

Hence, $f(x) = g(x)$, and so $f(x) = g(x) \forall x \in A \cap B$.

Suppose $f(x) = g(x)$ for all $x \in A \cap B$.

Suppose $x \in A \cup B$. So $x \in A$ or $x \in B$.

Suppose $x \in A$.

Then $(x, f(x)) \in f$, and so $(x, f(x)) \in f \cup g$.

Suppose $x \in B$.

Then $(x, g(x)) \in g$ and so $(x, g(x)) \in f \cup g$.

Thus, $\exists y \in C$ such that $(x, y) \in f \cup g$.

Suppose $(a, y_1) \in f \cup g$ and $(a, y_2) \in f \cup g$.

Suppose $(a, y_1) \in f$ and $(a, y_2) \in f$. Then $f(a) = y_1$ and $f(a) = y_2$, so $y_1 = y_2$.

Suppose $(a, y_1) \in g$ and $(a, y_2) \in g$. Then $g(a) = y_1$ and $g(a) = y_2$, so $y_1 = y_2$.

Suppose $(a, y_1) \in f$ and $(a, y_2) \in g$. Then $a \in A \cap B$, so $f(a) = y_1$, $g(a) = y_2$, and $f(a) = g(a)$, so $y_1 = y_2$.

Suppose $(a, y_1) \in g$ and $(a, y_2) \in f$. Then $a \in A \cap B$, so $f(a) = y_2$, $g(a) = y_1$, and $f(a) = g(a)$, so $y_1 = y_2$.

Hence, $y_1 = y_2$, and so, for each $a \in A \cup B$, there is a unique $y \in C$ such that $(a, y) \in f \cup g$.

Thus, $f \cup g : A \cup B \rightarrow C$. ■

3. Prove that $18 \mid 49^n + 6n - 1$ for all integers, $n \geq 1$.

For integers $n \geq 1$, let $P(n)$ be the statement " $18 \mid 49^n + 6n - 1$ ".

Base case:

Let $n = 1$.

Then $49^n + 6n - 1 = 49 + 6 - 1 = 54 = (3)(18)$, so $18 \mid 49^n + 6n - 1$, and so $P(1)$ is true.

Induction step:

Suppose $P(n)$ is true for some $n = k \geq 1$.

So $P(k)$ is true.

Then $18 \mid 49^k + 6k - 1$, so $49^k + 6k - 1 = 18m$ for some integer m .

Then:

$$\begin{aligned} 49^{k+1} + 6(k+1) - 1 &= 49(49^k + 6k - 1) - 49(6k) + 49 + 6(k+1) - 1 \\ &= 49(49^k + 6k - 1) - 48(6k) + 54 \\ &= 49(18m) - 18(16k) + (3)(18) \\ &= 18(49m - 16k + 3). \end{aligned}$$

Since m and k are integers, $49m - 16k + 3$ is an integer, and so $18 \mid 49^{k+1} + 6(k+1) - 1$.

Thus $P(k+1)$ is true, and so $P(k)$ implies $P(k+1)$.

Hence, since $P(1)$ is true, by induction $P(n)$ is true for all integers $n \geq 1$. ■

4. Let $\mathcal{F}$ and $\mathcal{G}$ be families (i.e., sets of sets).

(a) Prove that

$$\cup \mathcal{F} \setminus \cup \mathcal{G} \subseteq \cup (\mathcal{F} \setminus \mathcal{G}).$$

Suppose $x \in \cup \mathcal{F} \setminus \cup \mathcal{G}$.

Then $x \in \cup \mathcal{F}$ and $x \notin \cup \mathcal{G}$.

Hence $\exists A \in \mathcal{F}$ such that $x \in A$.

Since $x \notin \cup \mathcal{G}$, $A \notin \mathcal{G}$.

Hence, $A \in \mathcal{F} \setminus \mathcal{G}$, so $x \in \cup (\mathcal{F} \setminus \mathcal{G})$.

Thus, $x \in \cup \mathcal{F} \setminus \cup \mathcal{G}$ implies $x \in \cup (\mathcal{F} \setminus \mathcal{G})$, and so $\cup \mathcal{F} \setminus \cup \mathcal{G} \subseteq \cup (\mathcal{F} \setminus \mathcal{G})$. ■

(b) Give an example of non-empty families $\mathcal{F}$ and $\mathcal{G}$ such that

$$\cup \mathcal{F} \setminus \cup \mathcal{G} \neq \cup (\mathcal{F} \setminus \mathcal{G}).$$

Prove that your example is valid.

Let $\mathcal{F} = \{\{1, 2\}, \{2\}\}$ and $\mathcal{G} = \{\{2\}\}$.

Then

$$\cup \mathcal{F} \setminus \cup \mathcal{G} = \{1, 2\} \setminus \{2\} = \{1\}$$

and

$$\cup (\mathcal{F} \setminus \mathcal{G}) = \cup \{\{1, 2\}\} = \{1, 2\}.$$

Since $\{1\} \neq \{1, 2\}$, $\cup \mathcal{F} \setminus \cup \mathcal{G} \neq \cup (\mathcal{F} \setminus \mathcal{G})$. ■