

Math 300 C - Spring 2016  
Midterm Exam Number Two  
May 18, 2016

Name: \_\_\_\_\_

Student ID no. : \_\_\_\_\_

Signature: \_\_\_\_\_

|       |    |  |
|-------|----|--|
| 1     | 10 |  |
| 2     | 10 |  |
| 3     | 10 |  |
| 4     | 10 |  |
| Total | 40 |  |

- There are four problems.
- You have 50 minutes to complete the exam.

1. Let  $A = \mathbb{R} \times \mathbb{R} \setminus \{(0, 0)\}$ .

Thus,  $A$  is the  $xy$ -plane without the origin.

Define a relation  $R$  on  $A$  by

$((x_1, y_1), (x_2, y_2)) \in R \Leftrightarrow (x_1, y_1)$  and  $(x_2, y_2)$  lie on a line which passes through the origin.

Prove that  $R$  is transitive.

2. Let  $A$ ,  $B$ , and  $C$  be sets.

Let  $f : A \rightarrow C$  and  $g : B \rightarrow C$ .

Prove that  $f \cup g : A \cup B \rightarrow C$  iff  $f(x) = g(x)$  for all  $x \in A \cap B$ .

3. Prove that  $18 \mid 49^n + 6n - 1$  for all integers,  $n \geq 1$ .

4. Let  $\mathcal{F}$  and  $\mathcal{G}$  be families (i.e., sets of sets).

(a) Prove that

$$\cup \mathcal{F} \setminus \cup \mathcal{G} \subseteq \cup (\mathcal{F} \setminus \mathcal{G}).$$

(b) Give an example of non-empty families  $\mathcal{F}$  and  $\mathcal{G}$  such that

$$\cup \mathcal{F} \setminus \cup \mathcal{G} \neq \cup (\mathcal{F} \setminus \mathcal{G}).$$

Prove that your example is valid.

### Axioms of the Integers (AIs)

Suppose  $a$ ,  $b$ , and  $c$  are integers.

- **Closure:**

$a + b$  and  $ab$  are integers.

- **Substitution of Equals:**

If  $a = b$ , then  $a + c = b + c$  and  $ac = bc$ .

- **Commutativity:**

$a + b = b + a$  and  $ab = ba$ .

- **Associativity:**

$(a + b) + c = a + (b + c)$  and  $(ab)c = a(bc)$ .

- **The Distributive Law:**

$a(b + c) = ab + ac$

- **Identities:**

$a + 0 = 0 + a = a$  and  $a \cdot 1 = 1 \cdot a = a$

0 is called the *additive identity*

1 is called the *multiplicative identity*.

- **Additive Inverses:**

There exists an integer  $-a$  such that

$a + (-a) = (-a) + a = 0$ .

- **Trichotomy:**

Exactly one of the following is true:

$a < 0$ ,  $-a < 0$ , or  $a = 0$ .

### Sets

$A \subseteq B$  iff  $x \in A$  implies  $x \in B$

$A = B$  iff  $A \subseteq B$  and  $B \subseteq A$

$x \in A \cup B$  iff  $x \in A$  or  $x \in B$

$x \in A \cap B$  iff  $x \in A$  and  $x \in B$

$x \in A \setminus B$  iff  $x \in A$  and  $x \notin B$

$\mathcal{P}(A)$  is the set of all subsets of a set  $A$

### Elementary Properties of the Integers (EPIs)

Suppose  $a$ ,  $b$ ,  $c$ , and  $d$  are integers.

1.  $a \cdot 0 = 0$

2. If  $a + c = b + c$ , then  $a = b$ .

3.  $-a = (-1) \cdot a$

4.  $(-a) \cdot b = -(a \cdot b)$

5.  $(-a) \cdot (-b) = a \cdot b$

6. If  $a \cdot b = 0$ , then  $a = 0$  or  $b = 0$ .

7. If  $a \leq b$  and  $b \leq a$ , then  $a = b$ .

8. If  $a < b$  and  $b < c$ , then  $a < c$ .

9. If  $a < b$ , then  $a + c < b + c$ .

10. If  $a < b$  and  $0 < c$ , then  $ac < bc$ .

11. If  $a < b$  and  $c < 0$ , then  $bc < ac$ .

12. If  $a < b$  and  $c < d$ , then  $a + c < b + d$ .

13. If  $0 \leq a < b$  and  $0 \leq c < d$ , then  $ac < bd$ .

14. If  $a < b$ , then  $-b < -a$ .

15.  $0 \leq a^2$

16. If  $ab = 1$ , then either  $a = b = 1$  or  $a = b = -1$ .

NOTE: Properties 8-14 hold if each  $<$  is replaced with  $\leq$ .

One theorem for reference:

**Theorem DAS (Divisors are Smaller):** Let  $a$  and  $b$  be positive integers. Then  $a|b$  implies  $a \leq b$ .