

Math 300 C - Spring 2016
Midterm Exam Number One
April 20, 2016
Answers

1. **Theorem:** Suppose x and a are integers and $a > 0$.

For any integer y , define $|y|$ by

$$|y| = \begin{cases} y & \text{if } y \geq 0, \\ -y & \text{if } y < 0. \end{cases}$$

Then $|ax| = a|x|$.

Proof: Suppose x and a are integers and $a > 0$.

By the Trichotomy Law, $x > 0$, $x < 0$ or $x = 0$.

Suppose $x > 0$.

Then $|x| = x$, and $a|x| = ax$ by Substitution of Equals.

Also, $ax > 0$ by EPI 10 and 1, so $|ax| = ax$.

Hence, by Symmetry and Transitivity of Equals, $|ax| = a|x|$.

Suppose $x = 0$.

Then $|x| = |0| = 0$ and $a|x| = 0$ by EPI 1.

Also, $ax = 0$ by EPI 1 and so $|ax| = 0$.

Hence, by Symmetry and Transitivity of Equals, $|ax| = a|x|$.

Suppose $x < 0$.

Then $|x| = -x$ and $a|x| = a(-x)$ by Substitution of Equals.

Also, $ax < a \cdot 0$ by EPI 10, so $ax < 0$ by EPI 1.

So, $|ax| = -(ax)$.

By EPI 3, $-(ax) = (-1)ax$.

Also, $a(-x) = a(-1)x = (-1)ax$ by Commutativity and EPI 3.

Thus, by Symmetry and Transitivity of Equals, $|ax| = a|x|$.

Therefore, in all cases, $|ax| = a|x|$. ■

2. **Theorem:** Let A , B and C be sets.

Suppose $A \cup C \subseteq B \cup C$.

Then $A \setminus C \subseteq B$.

Proof: Let A , B and C be sets.

Suppose $A \cup C \subseteq B \cup C$.

Suppose $x \in A \setminus C$.

Then $x \in A$ and $x \notin C$.

Then $x \in A \cup C$, and so $x \in B \cup C$.

So $x \in B$ or $x \in C$.

Since $x \notin C$, $x \in B$.

Thus, $x \in A \setminus C$ implies $x \in B$,

$A \setminus C \subseteq B$. ■

3. **Theorem:** Let A, B and C be sets.

Suppose $A \subseteq B \cup C$ and $B \subseteq A \cup C$.

Then $A \cup B \cup C = C \cup (A \cap B)$.

Proof:

Let A, B and C be sets.

Suppose $A \subseteq B \cup C$ and $B \subseteq A \cup C$.

Suppose $x \in A \cup B \cup C$.

Then $x \in A$ or $x \in B$ or $x \in C$.

Suppose $x \in C$.

Then $x \in C \cup (A \cap B)$.

Suppose $x \in A$.

Then $x \in B \cup C$ (since $A \subseteq B \cup C$).

So $x \in B$ or $x \in C$.

Suppose $x \in C$.

Then $x \in C \cup (A \cap B)$.

Suppose $x \in B$.

Then $x \in A \cap B$, so $x \in C \cup (A \cap B)$.

Suppose $x \in B$.

Then $x \in A \cup C$ (since $B \subseteq A \cup C$).

So $x \in A$ or $x \in C$.

Suppose $x \in C$.

Then $x \in C \cup (A \cap B)$.

Suppose $x \in A$.

Then $x \in A \cap B$, so $x \in C \cup (A \cap B)$.

Hence, $x \in A \cup B \cup C$ implies $x \in C \cup (A \cap B)$, so $A \cup B \cup C \subseteq C \cup (A \cap B)$.

Suppose $x \in C \cup (A \cap B)$.

Then $x \in C$ or $x \in A \cap B$.

Suppose $x \in C$.

Then $x \in A \cup B \cup C$.

Suppose $x \in A \cap B$.

Then $x \in A$, so $x \in A \cup B \cup C$.

Hence, $x \in C \cup (A \cap B)$ implies $x \in A \cup B \cup C$, so $C \cup (A \cap B) \subseteq A \cup B \cup C$.

Thus, $A \cup B \cup C = C \cup (A \cap B)$. ■

4. **Theorem:** Let a and b be integers. Then $a^2 = b^2$ if and only if $a = b$ or $a = -b$.

Proof: Let a and b be integers.

We know from algebraic experience that $a^2 - b^2 = (a - b)(a + b)$, but we need to prove it here.

We have

$$\begin{aligned}(a - b)(a + b) &= (a - b)a + (a - b)b && \text{by Distributivity} \\ &= (a^2 - ba) + (ab - b^2) && \text{by Distributivity} \\ &= ((a^2 - ba) + ab) - b^2 && \text{by Associativity} \\ &= (a^2 + (-ba + ba)) - b^2 && \text{by Associativity and Commutativity} \\ &= (a^2 + 0) - b^2 && \text{by Additive Inverses} \\ &= a^2 - b^2 && \text{by Additive Identity.}\end{aligned}$$

So, by Transitivity of Equals, $(a - b)(a + b) = a^2 - b^2$.

Suppose $a^2 = b^2$.

Then $a^2 - b^2 = 0$ (by the Additive Inverses axiom), and so $(a - b)(a + b) = 0$ by Transitivity of Equals.

By EPI 6, $a - b = 0$ or $a + b = 0$.

Hence, $a = b$ or $a = -b$ by the Additive Inverses axiom and Substitution of Equals.

So $a^2 = b^2$ implies $a = b$ or $a = -b$.

Suppose $a = b$.

Then $a^2 = ba$ by Substitution of Equals.

Also, $ab = b^2$ by Substitution of Equals.

Hence, $a^2 = b^2$ by Commutativity and Transitivity of Equals.

Suppose $a = -b$.

Then $a^2 = -ba$ by Substitution of Equals.

Also, $a(-b) = (-b)^2$ by Substitution of equals.

We have $(-b)^2 = (-b)(-b) = b^2$ by EPI 5.

Also, $a(-b) = a(-1)b = (-1)ab = (-1)ba = -ab$ by Commutativity and EPI 3.

Hence, by Transitivity of Equals, $a^2 = b^2$.

Thus, $a^2 = b^2$ iff $a = b$ or $a = -b$. ■