

Math 300 C - Spring 2015
Midterm Exam Number Two
May 20, 2015
Solutions

1. Let $A = \mathbb{Z}_{\neq 0}$. So A is the set of all non-zero integers.

Define a relation $R \subseteq A \times A$ by

$$(x, y) \in R \text{ iff } xy > 0.$$

Prove that R is an equivalence relation.

Reflexive? Let $x \in A$.

Then $x > 0$ or $x < 0$, and so $x^2 > 0$.

Hence, $(x, x) \in R$.

Thus, R is reflexive.

Symmetric? Suppose $(x, y) \in R$.

Then $xy > 0$.

Since $yx = xy > 0$, $yx > 0$.

Hence, $(y, x) \in R$.

Thus $(x, y) \in R$ implies $(y, x) \in R$, so R is symmetric.

Transitive? Suppose $(x, y) \in R$ and $(y, z) \in R$.

Then $xy > 0$ and $yz > 0$.

Suppose $x > 0$. Then $y > 0$, and hence $z > 0$.

Thus $xz > 0$.

Suppose $x < 0$. Then $y < 0$, and hence $z < 0$.

Thus, $xz > 0$.

Hence, $xz > 0$, and so $(x, z) \in R$.

Thus $(x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R$.

Hence, R is transitive.

Since R is reflexive, symmetric, and transitive, R is an equivalence relation. ■

2. Suppose A , B and C are sets and $f : A \rightarrow B$ and $g : B \rightarrow C$.

Prove the following two statements.

(a) Suppose $g \circ f$ is onto. Then g is onto.

Suppose $y \in C$.

Then there exists $a \in A$ such that $g(f(a)) = y$.

Let $b = f(a)$.

Then $g(b) = y$.

Hence, for all $y \in C$, there exists a $b \in B$ such that $g(b) = y$.

Therefore, g is onto. ■

(b) Suppose g and f are one-to-one. Then $g \circ f$ is one-to-one.

Suppose $a_1 \in A$ and $a_2 \in A$, and suppose $g(f(a_1)) = g(f(a_2))$.

Since g is one-to-one, we can conclude that $f(a_1) = f(a_2)$.

Since f is one-to-one, we can conclude that $a_1 = a_2$.

Thus, $g(f(a_1)) = g(f(a_2))$ implies $a_1 = a_2$, and so $g \circ f$ is one-to-one. ■

3. Let A be a set.

Prove the following two statements.

(a) $A = \cup \mathcal{P}(A)$.

Suppose $x \in A$.

Since $A \subseteq A$, $A \in \mathcal{P}(A)$, and so $x \in \cup \mathcal{P}(A)$.

Thus $x \in A$ implies $x \in \cup \mathcal{P}(A)$, and so $A \subseteq \cup \mathcal{P}(A)$.

Now, suppose $x \in \cup \mathcal{P}(A)$.

Then there is a set $S \in \mathcal{P}(A)$ such that $x \in S$.

Then $S \subseteq A$, and, since $x \in S$, $x \in A$.

Thus, $x \in \cup \mathcal{P}(A)$ implies $x \in A$, and so $\cup \mathcal{P}(A) \subseteq A$.

Since $A \subseteq \cup \mathcal{P}(A)$ and $\cup \mathcal{P}(A) \subseteq A$, $A = \cup \mathcal{P}(A)$. ■

(b) Let $\mathcal{F}$ be a family of sets. If $\cup \mathcal{F} \subseteq A$, then $\mathcal{F} \subseteq \mathcal{P}(A)$.

Suppose $\cup \mathcal{F} \subseteq A$.

Suppose $S \in \mathcal{F}$.

Suppose $x \in S$.

Then, since $S \in \mathcal{F}$, $x \in \cup \mathcal{F}$.

Hence, $x \in A$.

Thus, $x \in S$ implies $x \in A$, so $S \subseteq A$, i.e., $S \in \mathcal{P}(A)$.

Thus, $S \in \mathcal{F}$ implies $S \in \mathcal{P}(A)$.

Therefore, $\mathcal{F} \subseteq \mathcal{P}(A)$. ■

4. Let $n \in \mathbb{Z}_{>0}$. Use induction to prove that

$$\sum_{i=1}^n i2^i = (n-1)2^{n+1} + 2.$$

Let $n \in \mathbb{Z}_{>0}$.

Define $P(n)$ to be the statement " $\sum_{i=1}^n i2^i = (n-1)2^{n+1} + 2$."

Base case: Suppose $n = 1$.

$$\text{Then } \sum_{i=1}^n i2^i = \sum_{i=1}^1 i2^i = (1)(2^1) = 2.$$

Also $(n-1)2^{n+1} + 2 = 2$, so $P(1)$ is true.

Induction step: Suppose $P(n)$ is true for some $n = x \in \mathbb{Z}_{>0}$.

Then $P(x)$ is true, so $\sum_{i=1}^x i2^i = (x-1)2^{x+1} + 2$.

Then:

$$\begin{aligned} \sum_{i=1}^{x+1} i2^i &= (x+1)2^{x+1} + \sum_{i=1}^x i2^i \\ &= (x+1)2^{x+1} + (x-1)2^{x+1} + 2 \\ &= 2^{x+1}(x+1+x-1) + 2 \\ &= 2x2^{x+1} + 2 \\ &= x2^{x+2} + 2 \\ &= ((x+1)-1)2^{(x+1)+1} + 2 \end{aligned}$$

so $P(x+1)$ is true.

Thus, $P(x)$ implies $P(x+1)$, and since $P(1)$ is true, by induction we have $P(n)$ is true for all $n \in \mathbb{Z}_{>0}$. ■

5. Prove that $11 \mid 4^{1404} - 3$.

We have the following congruences:

$$\begin{aligned} 4 &\equiv 4 \pmod{11} \\ 4^2 &\equiv 5 \pmod{11} \\ 4^3 &\equiv 9 \pmod{11} \\ 4^4 &\equiv 3 \pmod{11} \\ 4^5 &\equiv 1 \pmod{11} \end{aligned}$$

Then $4^{10} \equiv 1 \pmod{11}$, and so $4^{1400} = (4^{10})^{140} \equiv 1 \pmod{11}$.

Thus, $4^{1404} = 4^4 4^{1400} \equiv 3 \pmod{11}$, and so $11 \mid 4^{1404} - 3$. ■