1. (Short answer)

(a) Let $A$ be the set $\{a, b\}$. List every element of the power set of $A$.

$$\emptyset, \{a\}, \{b\}, \{a, b\}$$

(b) Use truth tables to show that $\neg P \lor Q$ and $\neg (P \land \neg Q)$ are equivalent.

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<th>$P$</th>
<th>$Q$</th>
<th>$\neg P$</th>
<th>$\neg P \lor Q$</th>
<th>$\neg Q$</th>
<th>$P \land \neg Q$</th>
<th>$\neg (P \land \neg Q)$</th>
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(c) Give a useful negation of this statement:

"For every integer $n$, there is an integer $m$ such that $m | n$.

One negation is: "There exists an integer $n$, such that, for all integers $m$, $m$ does not divide $n$.”

2. Prove that, for any sets $A$ and $B$,

$$\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B).$$

Proof. Suppose $x \in \mathcal{P}(A) \cup \mathcal{P}(B)$. Then $x \in \mathcal{P}(A)$ or $x \in \mathcal{P}(B)$.

Suppose $x \in \mathcal{P}(A)$.

Then $x \subseteq A$.

Suppose $y \in x$. Then $y \in A$, so $y \in A \cup B$. Hence, $x \subseteq A \cup B$.

Therefore, $x \in \mathcal{P}(A \cup B)$.

An identical argument shows that if $x \in \mathcal{P}(B)$, then $x \in \mathcal{P}(A \cup B)$.

Thus, if $x \in \mathcal{P}(A) \cup \mathcal{P}(B)$, then $x \in \mathcal{P}(A \cup B)$.

Therefore, $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$. $\square$

3. Prove that, for all integers $n$, $n^2 - 2$ is not divisible by 3. (HINT: Every integer can be written in the form $3k + r$, for $k$ and $r$ integers, and $r = 0, 1, \text{ or } 2$).
Proof. Let \( n \) be an integer. Then \( n = 3k + r \) for integers \( k \) and \( r \) with \( r = 0, 1 \) or \( 2 \).

Suppose \( r = 0 \). Then \( n = 3k \), so \( n^2 - 2 = 9k^2 - 2 \) and
\[
\frac{n^2 - 2}{3} = 3k^2 - \frac{2}{3}
\]
which is not an integer. So 3 does not divide \( n^2 - 2 \).

Suppose \( r = 1 \). Then \( n = 3k + 1 \), so \( n^2 - 2 = 9k^2 + 6k - 1 \) and
\[
\frac{n^2 - 2}{3} = 3k^2 + 2k - \frac{1}{3}
\]
which is not an integer. So 3 does not divide \( n^2 - 2 \).

Suppose \( r = 2 \). Then \( n = 3k + 2 \), so \( n^2 - 2 = 9k^2 + 12k + 2 \) and
\[
\frac{n^2 - 2}{3} = 3k^2 + 4k + \frac{2}{3}
\]
which is not an integer. So 3 does not divide \( n^2 - 2 \).

Thus, 3 does not divide \( n^2 - 2 \) for any integer \( n \). \( \square \)

4. Let \( n \) be an integer. Prove that \( 20 \mid n \) iff \( 4 \mid n \) and \( 5 \mid n \).

Proof. Let \( n \) be an integer.

Suppose \( 20 \mid n \). Then \( n = 20k \) for some integer \( k \). Hence \( n = 4(5k) \) and \( n = 5(4k) \) so \( 4 \mid n \) and \( 5 \mid n \).

Suppose \( 4 \mid n \) and \( 5 \mid n \). Then \( n = 4k \) and \( n = 5m \) for integers \( k \) and \( m \). Hence,
\[
n = 5n - 4n = 20k - 20m = 20(k - m)
\]

and, since \( k - m \) is an integer, \( 20 \mid n \).

Thus, \( 20 \mid n \) iff \( 4 \mid n \) and \( 5 \mid n \). \( \square \)

5. Prove that for all \( x \in \mathbb{R} \), there exists a \( y \in \mathbb{R} \) such that \( y \neq x \) and \( y^2 - x = x^2 - y \).

Proof. Let \( x \in \mathbb{R} \). Let \( y = -1 - x \). Then
\[
y^2 - x = (-1 - x)^2 - x = 1 + 2x + x^2 - x = 1 + x + x^2
\]
and
\[
x^2 - y = x^2 - (1 - x) = x^2 + x + 1
\]
so \( y^2 - x = x^2 - y \). \( \square \)