

Math 300 C, D - Autumn 2025
Midterm Exam Number Two
November 12, 2025
Solutions

1. (a) Prove the following theorem:

Theorem: Let $\mathcal{F}$ be a family.

Then $\mathcal{F} \subseteq \mathcal{P}(\bigcup \mathcal{F})$.

Proof:

Suppose $\mathcal{F}$ is a family.

Suppose $S \in \mathcal{F}$.

Suppose $x \in S$.

Then $x \in \bigcup \mathcal{F}$

So $x \in S$ implies $x \in \bigcup \mathcal{F}$, so $S \subseteq \bigcup \mathcal{F}$.

Hence, $S \in \mathcal{P}(\bigcup \mathcal{F})$.

Thus $S \in \mathcal{F}$ implies $S \in \mathcal{P}(\bigcup \mathcal{F})$, so $\mathcal{F} \subseteq \mathcal{P}(\bigcup \mathcal{F})$. ■

- (b) Give a set A and an equivalence relation $R \subseteq A \times A$ such that R has exactly six elements. (You need not show that R is an equivalence relation.)

There are two possible answer types.

- i. Let $A = \{1, 2, 3, 4, 5, 6\}$ and $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$.
- ii. Let $A = \{1, 2, 3, 4\}$ and $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 1)\}$.

[Note that for a set A with n elements, any equivalence relation on A must have $n + 2k$ elements where k is a non-negative integer. So if R is to have six elements, then we can conclude that A must have 2, 4 or 6 elements. However, if A has two elements, $A \times A$ has four elements, so R cannot have more than four elements. If A has four or six elements, then the only possible R 's are the ones with the same structure as the ones shown above.]

2. Prove the following theorem:

Theorem: Let A , B , and C be sets. Then

$$A \cap B \subseteq C \text{ and } A \cap C \subseteq B$$

if and only if

$$A \setminus B = A \setminus C.$$

Proof:

Let A , B , and C be sets.

Suppose $A \cap B \subseteq C$ and $A \cap C \subseteq B$.

Suppose $x \in A \setminus B$.

Then $x \in A$ and $x \notin B$.

Suppose $x \in C$.

Then $x \in A \cap C$.

Since $A \cap C \subseteq B$, $x \in B$.

This contradicts $x \notin B$, so $x \in C$ is false.

Hence, $x \notin C$, so $x \in A \setminus C$.

Thus, $x \in A \setminus B$ implies $x \in A \setminus C$.

Hence, $A \setminus B \subseteq A \setminus C$.

Now, suppose $x \in A \setminus C$.

Then $x \in A$ and $x \notin C$.

Suppose $x \in B$.

Then $x \in A \cap B$.

Since $A \cap B \subseteq C$, $x \in C$.

This contradicts $x \notin C$, so $x \in B$ is false.

Hence, $x \notin B$, so $x \in A \setminus B$.

Thus, $x \in A \setminus C$ implies $x \in A \setminus B$.

Hence, $A \setminus C \subseteq A \setminus B$.

Since $A \setminus C \subseteq A \setminus B$ and $A \setminus B \subseteq A \setminus C$, we conclude that $A \setminus B = A \setminus C$.

Suppose $A \setminus B = A \setminus C$.

Suppose $x \in A \cap B$.

Then $x \in A$ and $x \in B$.

Suppose $x \notin C$.

Then $x \in A \setminus C$, so $x \in A \setminus B$.

Since $x \in A$, we conclude $x \notin B$.

This contradicts $x \in B$.

Thus the supposition $x \notin C$ is false.

So, $x \in C$.

Hence, $x \in A \cap B$ implies $x \in C$.

Thus, $A \cap B \subseteq C$.

Suppose $x \in A \cap C$.

Then $x \in A$ and $x \in C$.

Suppose $x \notin B$.

Then $x \in A \setminus B$, so $x \in A \setminus C$.

Since $x \in A$, this implies $x \notin C$.

This contradicts $x \in C$.

Thus the supposition $x \notin B$ is false.

So, $x \in B$.

Hence, $x \in A \cap C$ implies $x \in B$.

Thus $A \cap C \subseteq B$. ■

3. Define a relation T on $\mathbb{R}$ by

$$(a, b) \in T \text{ iff there exists } n, m \in \mathbb{Z}_{>0} \text{ such that } a^m = b^n$$

Prove that T is an equivalence relation.

Proof:

Define T as above.

Let $x \in \mathbb{R}$.

Then $x^1 = x^1$, and $1 \in \mathbb{Z}_{>0}$, so $(x, x) \in T$.

Thus, for all $x \in \mathbb{R}$, $(x, x) \in T$, so T is **reflexive**.

Suppose $(x, y) \in T$.

Then there exist $a, b \in \mathbb{Z}_{>0}$ such that $x^a = y^b$.

So then $y^b = x^a$, and so $(y, x) \in T$.

Thus $(x, y) \in T$ implies $(y, x) \in T$, and so T is **symmetric**.

Suppose $(x, y) \in T$ and $(y, z) \in T$.

Then there exist $a, b, c, d \in \mathbb{Z}_{>0}$ such that $x^a = y^b$ and $y^c = z^d$.

Then $x^{ac} = y^{bc}$ and $y^{bc} = z^{bd}$, so $x^{ac} = z^{bd}$.

Since $ac \in \mathbb{Z}_{>0}$ and $bd \in \mathbb{Z}_{>0}$, we conclude that $(x, z) \in T$.

Thus $(x, y) \in T$ and $(y, z) \in T$ implies $(x, z) \in T$, and so T is **transitive**.

Since T is reflexive, symmetric and transitive, T is an equivalence relation. ■

4. Use induction to prove the following theorem.

Theorem: If $n \in \mathbb{Z}$ and $n \geq 6$, then $n! > n^3$.

Proof:

For n any integer, let $P(n)$ be the statement $n! > n^3$.

Base case: Let $n = 6$.

Then $n! = 6! = 720$ and $n^3 = 6^3 = 216$, and so we see $n! > n^3$.

So $P(6)$ is true.

Induction step:

Suppose $P(k)$ is true for some integer $k \geq 6$.

Then $k! > k^3$.

Then we have

$$\begin{aligned}(k+1)! &= (k+1)(k!) \\ &> (k+1)k^3 \\ &\geq (k+1)6k^2 \\ &= (k+1)(k^2 + 4k^2 + k^2) \\ &\geq (k+1)(k^2 + 24k + 36) \\ &> (k+1)(k^2 + 2k + 1) \\ &= (k+1)(k+1)^2 \\ &= (k+1)^3.\end{aligned}$$

Thus, $P(k+1)$ is true.

Hence, $P(k)$ implies $P(k+1)$.

This, together with the fact that $P(6)$ is true, allows us to say that, by induction, $P(n)$ is true for all integers $n \geq 6$. ■

Alternatively, we could argue like this:

$$\begin{aligned}(k+1)! &= (k+1)(k!) \\ &> (k+1)k^3 \\ &= k^4 + k^3 \\ &\geq 6k^3 + k^3 \\ &= 7k^3 \\ &= k^3 + 6k^3 \\ &\geq k^3 + 36k^2 \\ &= k^3 + 3k^2 + 33k^2 \\ &\geq k^3 + 3k^2 + 198k \\ &= k^3 + 3k^2 + 3k + 195k \\ &= k^3 + 3k^2 + 3k + 1170 \\ &> k^3 + 3k^2 + 3k + 1 \\ &= (k+1)^3.\end{aligned}$$

Elementary Properties of the Integers (EPIs)

Suppose a, b, c , and d are integers.

- **Closure:** $a + b$ and ab are integers.
- **Substitution of Equals:** If $a = b$, then $a + c = b + c$ and $ac = bc$.
- **Commutativity:** $a + b = b + a$ and $ab = ba$.
- **Associativity:** $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$.
- **The Distributive Law:** $a(b + c) = ab + ac$
- **Identities:** $a + 0 = 0 + a = a$ and $a \cdot 1 = 1 \cdot a = a$.
0 is called the additive identity.
1 is called the multiplicative identity.
- **Additive Inverses:** There exists an integer $-a$ such that $a + (-a) = (-a) + a = 0$.
- **Trichotomy:** Exactly one of the following is true:
 $a > 0$, $-a > 0$, or $a = 0$.
- **The Well-Ordering Principle:** Every non-empty set of positive integers contains a smallest element.

1. $a \cdot 0 = 0$
2. If $a + c = b + c$, then $a = b$.
3. $-a = (-1) \cdot a$
4. $(-a) \cdot b = -(ab)$
5. $(-a) \cdot (-b) = ab$
6. If $ab = 0$, then $a = 0$ or $b = 0$.
7. If $a \leq b$ and $b \leq a$, then $a = b$.
8. If $a < b$ and $b < c$, then $a < c$.
9. If $a < b$, then $a + c < b + c$.
10. If $a < b$ and $0 < c$, then $ac < bc$.
11. If $a < b$ and $c < 0$, then $bc < ac$.
12. If $a < b$ and $c < d$, then $a + c < b + d$.
13. If $0 \leq a < b$ and $0 \leq c < d$, then $ac < bd$.
14. If $a < b$, then $-b < -a$.
15. $0 \leq a^2$, where $a^2 = a \cdot a$.
16. If $ab = 1$, then either $a = b = 1$ or $a = b = -1$.

Sets

$A \subseteq B$ iff $x \in A$ implies $x \in B$

$A = B$ iff $A \subseteq B$ and $B \subseteq A$

$x \in A \cup B$ iff $x \in A$ or $x \in B$

$x \in A \cap B$ iff $x \in A$ and $x \in B$

$x \in A \setminus B$ iff $x \in A$ and $x \notin B$

$\mathcal{P}(A)$ is the set of all subsets of a set A

For a family $\mathcal{F}$:

$$\bigcup \mathcal{F} = \{x : \exists S \in \mathcal{F} \text{ such that } x \in S\}$$

$$\bigcap \mathcal{F} = \{x : \forall S \in \mathcal{F}, x \in S\}$$

NOTE: Properties 8-14 hold if each $<$ is replaced with $\leq$.