

Math 300 C, D - Autumn 2025
Midterm Exam Number Two
November 12, 2025

Name: _____

Student ID no. : _____

Signature: _____

1	10
2	10
3	10
4	10
Total	40

- You have 50 minutes to complete the exam.
- When time is called, you must immediately stop writing.
- No calculators or other devices are allowed during the exam.
- No notesheets or other information sources are allowed during the exam.
- The last page of the exam lists the EPIs and set facts for your use.
- You should not use any result from homework, etc., without first proving it on this exam, unless otherwise indicated.

1. (a) Prove the following theorem:

Theorem: Let $\mathcal{F}$ be a family.

Then $\mathcal{F} \subseteq \mathcal{P}(\bigcup F)$.

- (b) Give a set A and an equivalence relation $R \subseteq A \times A$ such that R has exactly six elements. (You need not show that R is an equivalence relation.)

2. Prove the following theorem:

Theorem: Let A , B , and C be sets. Then

$$A \cap B \subseteq C \text{ and } A \cap C \subseteq B$$

if and only if

$$A \setminus B = A \setminus C.$$

3. Define a relation T on $\mathbb{R}$ by

$$(a, b) \in T \text{ iff there exists } n, m \in \mathbb{Z}_{>0} \text{ such that } a^m = b^n$$

Prove that T is an equivalence relation.

4. Use induction to prove the following theorem.

Theorem: If $n \in \mathbb{Z}$ and $n \geq 6$, then $n! > n^3$.

Elementary Properties of the Integers (EPIs)

Suppose a, b, c , and d are integers.

- **Closure:** $a + b$ and ab are integers.
- **Substitution of Equals:** If $a = b$, then $a + c = b + c$ and $ac = bc$.
- **Commutativity:** $a + b = b + a$ and $ab = ba$.
- **Associativity:** $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$.
- **The Distributive Law:** $a(b + c) = ab + ac$
- **Identities:** $a + 0 = 0 + a = a$ and $a \cdot 1 = 1 \cdot a = a$.
0 is called the additive identity.
1 is called the multiplicative identity.
- **Additive Inverses:** There exists an integer $-a$ such that $a + (-a) = (-a) + a = 0$.
- **Trichotomy:** Exactly one of the following is true:
 $a > 0$, $-a > 0$, or $a = 0$.
- **The Well-Ordering Principle:** Every non-empty set of positive integers contains a smallest element.

1. $a \cdot 0 = 0$
2. If $a + c = b + c$, then $a = b$.
3. $-a = (-1) \cdot a$
4. $(-a) \cdot b = -(ab)$
5. $(-a) \cdot (-b) = ab$
6. If $ab = 0$, then $a = 0$ or $b = 0$.
7. If $a \leq b$ and $b \leq a$, then $a = b$.
8. If $a < b$ and $b < c$, then $a < c$.
9. If $a < b$, then $a + c < b + c$.
10. If $a < b$ and $0 < c$, then $ac < bc$.
11. If $a < b$ and $c < 0$, then $bc < ac$.
12. If $a < b$ and $c < d$, then $a + c < b + d$.
13. If $0 \leq a < b$ and $0 \leq c < d$, then $ac < bd$.
14. If $a < b$, then $-b < -a$.
15. $0 \leq a^2$, where $a^2 = a \cdot a$.
16. If $ab = 1$, then either $a = b = 1$ or $a = b = -1$.

Sets

$A \subseteq B$ iff $x \in A$ implies $x \in B$

$A = B$ iff $A \subseteq B$ and $B \subseteq A$

$x \in A \cup B$ iff $x \in A$ or $x \in B$

$x \in A \cap B$ iff $x \in A$ and $x \in B$

$x \in A \setminus B$ iff $x \in A$ and $x \notin B$

$\mathcal{P}(A)$ is the set of all subsets of a set A

For a family $\mathcal{F}$:

$$\bigcup \mathcal{F} = \{x : \exists S \in \mathcal{F} \text{ such that } x \in S\}$$

$$\bigcap \mathcal{F} = \{x : \forall S \in \mathcal{F}, x \in S\}$$

NOTE: Properties 8-14 hold if each $<$ is replaced with $\leq$.