

Math 300 C, D - Autumn 2025
Midterm Exam Number One
October 15, 2025
Solutions

1. *Using the Elementary Properties of the Integers, carefully prove the following theorem.*

Theorem: Let $a, b \in \mathbb{Z}$.

Then $a^2 - b^2 = (a + b)(a - b)$.

Proof: Let $a, b \in \mathbb{Z}$.

Then

$$\begin{aligned}(a + b)(a - b) &= (a + b)a + (a + b)(-b) && \text{(Distributive Law)} \\ &= a(a + b) + (-b)(a + b) && \text{(Commutativity, SOE)} \\ &= (a^2 + ab) + ((-b)a + (-b)b) && \text{(Distributive Law, SOE)} \\ &= (a^2 + ab) + (-ab) + (-b^2) && \text{(EPI 4, SOE)} \\ &= ((a^2 + ab) + (-ab)) + (-b^2) && \text{(Associativity)} \\ &= ((a^2 + (ab + (-ab))) + (-b^2) && \text{(Associativity, SOE)} \\ &= ((a^2 + 0) + (-b^2) && \text{(Additive Inverses, SOE)} \\ &= a^2 - b^2. && \text{(Additive Identity)}\end{aligned}$$

Hence, by Transitivity and Symmetry of Equals, $a^2 - b^2 = (a + b)(a - b)$. ■

2. Prove the following theorem.

Theorem: Let a and b be integers. Then $3a + 5b$ is odd iff a and b have opposite parity.

Proof: Let $a, b \in \mathbb{Z}$.

Suppose a and b have opposite parity.

Then one of a and b is odd and one is even.

Suppose a is odd and b is even.

Then $a = 2k + 1$ and $b = 2m$ for some $k, m \in \mathbb{Z}$.

So, $3a + 5b = 6k + 3 + 10m = 2(3k + 5m + 1) + 1$.

Since $3k + 5m + 1$ is an integer by closure, $3a + 5b$ is odd.

Suppose a is even and b is odd.

Then $a = 2k$ and $b = 2m + 1$ for some $k, m \in \mathbb{Z}$.

So, $3a + 5b = 6k + 10 + 5 = 2(3k + 5m + 2) + 1$.

Since $3k + 5m + 2$ is an integer by closure, $3a + 5b$ is odd.

Now, suppose a and b have the same parity.

Suppose a and b are even.

Then $a = 2k$ and $b = 2m$ for some $k, m \in \mathbb{Z}$.

So, $3a + 5b = 6k + 10m = 2(3k + 5m)$.

Since $3k + 5m$ is an integer by closure, $3a + 5b$ is even.

Suppose a and b are odd.

Then $a = 2k + 1$ and $b = 2m + 1$ for some $k, m \in \mathbb{Z}$.

So, $3a + 5b = 6k + 10m + 8 = 2(3k + 5m + 4)$.

Since $3k + 5m + 4$ is an integer by closure, $3a + 5b$ is even. ■

3. Give an example of an irrational number x such that x^2 is an integer that is divisible by 9.

Prove that you are correct.

(Note: you may use the fact that $\sqrt{2}$ is irrational, but you need to prove everything else you need.)

Let $x = 3\sqrt{2}$.

Then $x^2 = 18 = 2(9)$ so x^2 is an integer divisible by 9.

To show x is irrational, suppose x is rational.

Then $x = \frac{a}{b}$ for some integers a and b , with $b \neq 0$.

Then

$$3\sqrt{2} = \frac{a}{b}$$

so

$$\left(\frac{1}{3}\right)(3)\sqrt{2} = \frac{a}{b}\left(\frac{1}{3}\right)$$

and hence

$$\sqrt{2} = \frac{a}{3b}.$$

Since a and $3b$ are integers (by closure), and $3b \neq 0$ since $b \neq 0$, $\frac{a}{3b}$ is a rational number.

This is a contradiction, since $\sqrt{2}$ is irrational.

Hence, x is irrational. ■

4. Prove the following theorem.

Theorem: Let $a, b, c \in \mathbb{Z}$. If $a|bc$ and $a|(b + c)$, then $a|b^2$.

Proof: Let $a, b, c \in \mathbb{Z}$.

Suppose $a|bc$ and $a|(b + c)$.

Then $bc = ak$ and $b + c = am$ for some $k, m \in \mathbb{Z}$.

Hence,

$$c = am - b$$

so

$$bc = b(am - b) = ak$$

and so

$$abm - b^2 = ak.$$

Thus,

$$b^2 = a(bm - k).$$

Since $bm - k$ is an integer by closure, $a|b^2$. ■