

Math 300 C,D - Autumn 2024
Midterm Exam Number Two
November 15, 2024
Solutions

1. Prove the following theorem.

Theorem: Let A and B be sets. Then $A \cap B = \emptyset$ if and only if $\mathcal{P}(A) \cap \mathcal{P}(B) = \{\emptyset\}$.

Proof:

Let A and B be sets.

Suppose $A \cap B \neq \emptyset$.

Then there exists $x \in A \cap B$; so $x \in A$ and $x \in B$.

Let $S = \{x\}$.

Since $x \in A$, $S \subseteq A$, so $S \in \mathcal{P}(A)$.

Since $x \in B$, $S \subseteq B$, so $S \in \mathcal{P}(B)$.

Thus, $S \in \mathcal{P}(A) \cap \mathcal{P}(B)$, and $S \neq \emptyset$, so $\mathcal{P}(A) \cap \mathcal{P}(B) \neq \{\emptyset\}$.

Suppose $\mathcal{P}(A) \cap \mathcal{P}(B) \neq \{\emptyset\}$.

Since $\emptyset \in \mathcal{P}(A) \cap \mathcal{P}(B)$, we can conclude that there is a set $S \in \mathcal{P}(A) \cap \mathcal{P}(B)$ and $S \neq \emptyset$.

Hence, there is an element $x \in S$.

Since $S \in \mathcal{P}(A)$, $S \subseteq A$, and so $x \in A$.

Since $S \in \mathcal{P}(B)$, $S \subseteq B$, and so $x \in B$.

Thus, $x \in A \cap B$, so $A \cap B \neq \emptyset$.

Hence, $A \cap B = \emptyset$ if and only if $\mathcal{P}(A) \cap \mathcal{P}(B) = \{\emptyset\}$. ■

2. Prove the following theorem.

Theorem: Let A , B and C be sets. Then

$$(A \cap B) \setminus C = (A \setminus C) \cap (B \setminus C).$$

Proof: Let A , B and C be sets.

Suppose $x \in (A \cap B) \setminus C$.

Then $x \in A \cap B$ and $x \notin C$.

So $x \in A$ and $x \in B$.

Hence, $x \in A \setminus C$ and $x \in B \setminus C$.

Thus $x \in (A \setminus C) \cap (B \setminus C)$.

Therefore, $x \in (A \cap B) \setminus C$ implies $x \in (A \setminus C) \cap (B \setminus C)$ and hence $(A \cap B) \setminus C \subseteq (A \setminus C) \cap (B \setminus C)$.

Now, suppose $x \in (A \setminus C) \cap (B \setminus C)$.

Then $x \in A \setminus C$ and $x \in B \setminus C$.

So $x \in A$, $x \in B$ and $x \notin C$.

Hence, $x \in (A \cap B)$, so $x \in (A \cap B) \setminus C$.

Therefore, $x \in (A \setminus C) \cap (B \setminus C)$ implies $x \in (A \cap B) \setminus C$, and so $(A \setminus C) \cap (B \setminus C) \subseteq (A \cap B) \setminus C$.

Thus, $(A \cap B) \setminus C = (A \setminus C) \cap (B \setminus C)$. ■

3. Let $\mathcal{F}$ and $\mathcal{G}$ be families. Prove that if $\mathcal{F} \subseteq \mathcal{G}$, then $\bigcup \mathcal{F} \subseteq \bigcup \mathcal{G}$.

Proof:

Let $\mathcal{F}$ and $\mathcal{G}$ be families.

Suppose $x \in \bigcup \mathcal{F}$.

Then there exists $S \in \mathcal{F}$ such that $x \in S$.

Since $\mathcal{F} \subseteq \mathcal{G}$, $S \in \mathcal{G}$.

Since $S \in \mathcal{G}$ and $x \in S$, $x \in \bigcup \mathcal{G}$.

Thus, $x \in \bigcup \mathcal{F}$ implies $x \in \bigcup \mathcal{G}$.

Hence $\bigcup \mathcal{F} \subseteq \bigcup \mathcal{G}$. ■

4. Use induction to prove the following theorem.

Theorem: For all integers $n \geq 4$, $n! > n^2$.

Proof: Let $P(n) = "n! > n^2"$. We will prove $P(n)$ is true for all integers $n \geq 4$ using induction.

Base case: Let $n = 4$.

Then $n! = 24 > 16 = n^2$, so $P(4)$ is true.

Induction step: Suppose $P(n)$ is true for some integer $n = x \geq 4$.

So $P(x)$ is true.

Then

$$\begin{aligned}(x+1)! &= (x+1)(x!) \\ &> (x+1)x^2 \quad (\text{by our induction hypothesis } (P(x) \text{ is true})) \\ &\geq 4x(x+1) \quad (\text{since } x \geq 4) \\ &= (3x+x)(x+1) \\ &> (x+1)(x+1) \quad (\text{again, since } x \geq 4) \\ &= (x+1)^2.\end{aligned}$$

Thus, $P(x+1)$ is true.

Hence, $P(x)$ implies $P(x+1)$, and since $P(4)$ is true, by induction we conclude that $P(n)$ is true for all integers $n \geq 4$. ■

5. Let m be a positive integer. Define the relation $R \subseteq \mathbb{Z} \times \mathbb{Z}$ by

$$(x, y) \in R \text{ iff } m \mid (x - y).$$

Is R reflexive?

Is R symmetric?

Is R transitive?

Is R an equivalence relation?

(Note you have **four** questions to answer here.)

Let $x \in \mathbb{Z}$.

Then $x - x = 0 = 0 \cdot m$, so $m \mid x - x$.

Hence, $(x, x) \in R$.

So, $x \in \mathbb{Z}$ implies $(x, x) \in R$, and so R is reflexive.

Suppose $(x, y) \in R$.

Then $m \mid x - y$, so $x - y = mk$ for some integer k .

Then $y - x = -(x - y) = -mk = m(-k)$, and so $m \mid y - x$.

Hence, $(y, x) \in R$.

Thus, $(x, y) \in R$ implies $(y, x) \in R$, so R is symmetric.

Suppose $(x, y) \in R$ and $(y, z) \in R$.

Then $m \mid x - y$ and $m \mid y - z$.

Hence, there exist integers k and j such that $x - y = mk$ and $y - z = mj$.

Then $(x - y) + (y - z) = x - z = mk + mj = m(k + j)$, so $m \mid x - z$.

Hence, $(x, z) \in R$.

Thus, $(x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R$, so R is transitive.

Since R is reflexive, symmetric and transitive, R is an equivalence relation.