

Math 300 C,D - Autumn 2024
Midterm Exam Number One
October 16, 2024
Solutions

1. Write useful contrapositives of the following sentences. Express the contrapositives as sentences, not symbolically.

(a) If n is divisible by 2, 3, and 5, then n is divisible by 30.

If n is not divisible by 30, then n is not divisible by 2, not divisible by 3 or not divisible by 5.

(b) If the weather is warm and rainy, then I will sleep or watch a movie.

If I don't sleep and I don't watch a movie, then the weather is not warm or not rainy.

(c) If n is an integer, and n^2 is greater than 90 and less than 105, then $n = 10$.

If $n \neq 10$, then n is not an integer or $n^2 \leq 90$ or $n^2 \geq 105$.

2. Prove the following theorem.

Theorem: Let $a, b \in \mathbb{Z}$. Suppose $a < 0$. Then

$$|ab| = |a||b|.$$

Proof: Let $a, b \in \mathbb{Z}$.

Suppose $a < 0$.

We know $b < 0$ or $b \geq 0$ so we treat these two cases.

Case 1: Suppose $b < 0$.

Then $|a| = -a$ and $|b| = -b$.

Also, $ab > 0$, so $|ab| = ab$.

Hence, $|ab| = ab = (-a)(-b) = |a||b|$.

Case 2: Suppose $b \geq 0$.

Then $|a| = -a$ and $|b| = b$.

Also, $ab \leq 0$.

Suppose $ab < 0$.

Then $|ab| = -ab = (-a)(b) = |a||b|$.

Suppose $ab = 0$.

Then $|ab| = 0 = -0 = -ab = (-a)(b) = |a||b|$. ■

3. Prove the following theorem.

Theorem: *The product of two distinct irrational numbers may be a rational number. (HINT: You may use as examples any numbers we've proved are irrational, and also the fact that the product of an irrational number and a non-zero rational number is irrational.)*

Proof:

We know $\sqrt{2}$ is irrational.

Since $\frac{1}{3}$ is rational, and the product of an irrational number and a non-zero rational number is irrational, we conclude that

$$\frac{\sqrt{2}}{3}$$

is irrational.

Finally, we see the product of $\sqrt{2}$ and $\frac{\sqrt{2}}{3}$ is rational:

$$\sqrt{2} \left(\frac{\sqrt{2}}{3} \right) = \frac{2}{3}. \blacksquare$$

Commentary: We could use any irrational and any rational other than 0 or 1 here.

4. Prove the following theorem.

Theorem: Let n be an integer. Then $n^2 + 6n$ is divisible by 4 iff n is even.

Proof: Let n be an integer.

($\Leftarrow$) Suppose n is even.

Then $n = 2k$ for some integer k .

Then $n^2 + 6n = 4k^2 + 12k = 4(k^2 + 3k)$.

Since $k^2 + 3k \in \mathbb{Z}$, $4|n^2 + 6n$.

($\Rightarrow$) (by contrapositive)

Suppose n is odd.

Then $n = 2k + 1$ for some $k \in \mathbb{Z}$.

Then

$$\begin{aligned} n^2 + 6n &= (2k + 1)^2 + 6(2k + 1) \\ &= 4k^2 + 4k + 1 + 12k + 6 \\ &= 4k^2 + 16k + 4 + 3 \\ &= 4(k^2 + 4k + 1) + 3. \end{aligned}$$

Suppose $4|n^2 + 6n$.

Then $n^2 + 6n = 4(k^2 + 4k + 1) + 3 = 4r$ for some $r \in \mathbb{Z}$.

Then $3 = 4r - 4(k^2 + 4k + 1) = 4(r - (k^2 + 4k + 1))$.

So $4|3$. This is a contradiction, since 4 does not divide 3.

Hence $4 \nmid n^2 + 6n$. ■

Commentary: Alternatively, in the second part of the proof, we might argue that $n^2 + 6n = 4(k^2 + 4k + 1) + 3$ is odd, and also that if 4 divides $n^2 + 6n$ then $n^2 + 6n$ would be even, and hence 4 does not divide $n^2 + 6n$.

5. Prove the following theorem.

Theorem: Let $a, b \in \mathbb{Z}$. If $a|b$ and $(a+1)|b$, then $a(a+1)|b$.

Proof:

Let $a, b \in \mathbb{Z}$.

Suppose $a|b$ and $(a+1)|b$.

Then $b = ak$ for some $k \in \mathbb{Z}$

and $b = (a+1)m$ for some $m \in \mathbb{Z}$.

Then $ak = am + m$ so $m = a(k - m)$ (so $a|m$).

Thus, $b = (a+1)(a(k - m)) = (a(a+1))(k - m)$ and so $a(a+1)|b$. ■

Elementary Properties of the Integers (EPIs)

Suppose a, b, c , and d are integers.

- **Closure:** $a + b$ and ab are integers.
- **Substitution of Equals:** If $a = b$, then $a + c = b + c$ and $ac = bc$.
- **Commutativity:** $a + b = b + a$ and $ab = ba$.
- **Associativity:** $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$.
- **The Distributive Law:** $a(b + c) = ab + ac$
- **Identities:** $a + 0 = 0 + a = a$ and $a \cdot 1 = 1 \cdot a = a$.
 0 is called the additive identity.
 1 is called the multiplicative identity.
- **Additive Inverses:** There exists an integer $-a$ such that $a + (-a) = (-a) + a = 0$.
- **Trichotomy:** Exactly one of the following is true:
 $a > 0$, $-a > 0$, or $a = 0$.
- **The Well-Ordering Principle:** Every non-empty set of positive integers contains a smallest element.

1. $a \cdot 0 = 0$
2. If $a + c = b + c$, then $a = b$.
3. $-a = (-1) \cdot a$
4. $(-a) \cdot b = -(ab)$
5. $(-a) \cdot (-b) = ab$
6. If $ab = 0$, then $a = 0$ or $b = 0$.
7. If $a \leq b$ and $b \leq a$, then $a = b$.
8. If $a < b$ and $b < c$, then $a < c$.
9. If $a < b$, then $a + c < b + c$.
10. If $a < b$ and $0 < c$, then $ac < bc$.
11. If $a < b$ and $c < 0$, then $bc < ac$.
12. If $a < b$ and $c < d$, then $a + c < b + d$.
13. If $0 \leq a < b$ and $0 \leq c < d$, then $ac < bd$.
14. If $a < b$, then $-b < -a$.
15. $0 \leq a^2$, where $a^2 = a \cdot a$.
16. If $ab = 1$, then either $a = b = 1$ or $a = b = -1$.

NOTE: Properties 8-14 hold if each $<$ is replaced with $\leq$.