

Math 300 D - Autumn 2014

Midterm Exam Number One

October 15, 2014

Solutions

1. Use a truth table to decide whether the following argument is valid. Explain your conclusion.

- *Angela will be there, and Boris or Chen will be there.*
- *Angela will not be there, or Chen will not be there.*
- *Therefore, Boris and Chen will be there.*

Let A stand for “Angela will be there”, B for “Boris will be there” and C for “Chen will be there”.

The argument has the following truth table:

A	B	C	$A \wedge (B \vee C)$	$\neg A \vee \neg C$	$B \wedge C$
F	F	F	F	T	F
F	F	T	F	T	F
F	T	F	F	T	F
F	T	T	F	T	T
T	F	F	F	T	F
T	F	T	T	F	F
T	T	F	T	T	F
T	T	T	T	F	T

The second-to-last line of the table shows that the premises can both be true while the conclusion is false. Hence, the argument is not valid.

2. Let A, B , and C be sets. Verify the following identities by showing that that the statement $x \in$ “the left hand set” is equivalent to the statement $x \in$ “the right hand set”. Give a justification for each step (e.g., definition of union, DeMorgan’s law, distributive law, etc.)

Note: for problems of this sort, there are many correct answers. You should work out your own attempt before reading any of this.

(a) $(A \cap (B \cup C)) \cap C = A \cap C$

$$\begin{aligned}
 & x \in (A \cap (B \cup C)) \cap C \\
 \Leftrightarrow & (x \in A) \wedge (x \in B \cup C) \wedge (x \in C) && \text{(def. of } \cap \text{)} \\
 \Leftrightarrow & (x \in A) \wedge ((x \in B \vee x \in C) \wedge x \in C) && \text{(def. of union, assoc. law)} \\
 \Leftrightarrow & x \in A \wedge x \in C && \text{(absorption law)} \\
 \Leftrightarrow & x \in A \cap C && \text{(def. of } \cap \text{)}
 \end{aligned}$$

$$(b) (A \cup B) \setminus (A \cup C) = B \setminus (A \cup C)$$

$$\begin{aligned}
 & x \in (A \cup B) \setminus (A \cup C) \\
 \Leftrightarrow & (x \in A \vee x \in B) \wedge x \notin A \cup C && (\text{def. of } \cap, \text{ set difference}) \\
 \Leftrightarrow & (x \in A \vee x \in B) \wedge \neg(x \in A \vee x \in C) && (\text{def. of } \notin) \\
 \Leftrightarrow & (x \in A \vee x \in B) \wedge (x \notin A) \wedge x \notin C && (\text{DeMorgan's law}) \\
 \Leftrightarrow & ((x \in A \wedge x \notin A) \vee (x \in B \wedge x \notin A)) \wedge x \notin C && (\text{dist., assoc.}) \\
 \Leftrightarrow & x \in B \wedge x \notin A \wedge x \notin C && (\text{contradiction law, assoc.}) \\
 \Leftrightarrow & x \in B \wedge \neg(x \in A \vee x \in C) && (\text{DeMorgan's}) \\
 \Leftrightarrow & x \in B \wedge x \notin (A \cup C) && (\text{def. of } \notin, \cup) \\
 \Leftrightarrow & x \in B \setminus (A \cup C) && (\text{def. of set difference})
 \end{aligned}$$

3. Simplify the following expressions as much as possible. You should show a sequence of equivalent expressions connecting the original expression with your final one, and give a justification for each step (e.g., DeMorgan's law, distributive law, etc.)

Note: for problems of this sort, there are many correct answers. You should work out your own attempt before reading any of this.

$$(a) P \vee (Q \wedge (\neg P \vee R))$$

$$\begin{aligned}
 & P \vee (Q \wedge (\neg P \vee R)) \\
 \Leftrightarrow & (P \vee Q) \wedge (P \vee (\neg P \vee R)) && (\text{distrib.}) \\
 \Leftrightarrow & (P \vee Q) \wedge ((P \vee \neg P) \vee R) && (\text{assoc.}) \\
 \Leftrightarrow & P \vee Q && (\text{tautology law})
 \end{aligned}$$

$$(b) (P \vee Q) \wedge \neg(P \vee (Q \wedge \neg P))$$

$$\begin{aligned}
 & (P \vee Q) \wedge \neg(P \vee (Q \wedge \neg P)) \\
 \Leftrightarrow & (P \vee Q) \wedge \neg((P \vee Q) \wedge (P \vee \neg P)) && (\text{distrib.}) \\
 \Leftrightarrow & (P \vee Q) \wedge \neg(P \vee Q) && (\text{tautology law}) \\
 \Leftrightarrow & \text{contradiction} && (\text{def. of contradiction})
 \end{aligned}$$

4. Simplify the following expressions as much as possible. You should show a sequence of equivalent expressions connecting the original expression with your final one, and give a justification for each step (e.g., DeMorgan's law, distributive law, etc.)

$$(a) (R \wedge Q) \vee (P \rightarrow (\neg Q \wedge R))$$

$$\begin{aligned}
 & (R \wedge Q) \vee (P \rightarrow (\neg Q \wedge R)) \\
 \Leftrightarrow & (R \wedge Q) \vee (\neg P \vee (\neg Q \wedge R)) && (\text{conditional law}) \\
 \Leftrightarrow & ((R \wedge Q) \vee (R \wedge \neg Q)) \vee \neg P && (\text{assoc., commut.}) \\
 \Leftrightarrow & (R \wedge (Q \vee \neg Q)) \vee \neg P && (\text{distrib.}) \\
 \Leftrightarrow & R \vee \neg P && (\text{tautology law})
 \end{aligned}$$

$$(b) P \leftrightarrow (Q \rightarrow (P \rightarrow Q))$$

$$\begin{aligned}
 & P \leftrightarrow (Q \rightarrow (P \rightarrow Q)) \\
 \Leftrightarrow & P \leftrightarrow (\neg Q \vee (\neg P \vee Q)) && \text{(conditional law)} \\
 \Leftrightarrow & P \leftrightarrow ((\neg Q \vee Q) \vee \neg P) && \text{(associative law)} \\
 \Leftrightarrow & P \leftrightarrow T && \text{(tautology law)} \\
 \Leftrightarrow & (P \rightarrow T) \wedge (T \rightarrow P) && \text{(def. of } \leftrightarrow) \\
 \Leftrightarrow & (\neg P \vee T) \wedge (\text{Contra.} \vee P) && \text{(conditional law)} \\
 \Leftrightarrow & T \wedge P && \text{(tautology and contradiction laws)} \\
 \Leftrightarrow & P && \text{(tautology law)}
 \end{aligned}$$

5. Write useful contrapositives of the following sentences. Express the contrapositives as sentences.

(a) *If the sum of two integers is greater than 100, then at least one of the integers is greater than 25.*

If two integers are both less than or equal to 25, then their sum is less than or equal to 100.

(b) *If the morning sky is blue or pink, then the day is warm and pleasant.*

If the day is not warm or not pleasant, then the morning sky is not blue and is it not pink.

(c) *If x and y are integers, and at least one of them is zero, then $xy = 0$.*

If $xy \neq 0$, then x or y is not an integer, or both of them are non-zero.