

Math 300 D - Autumn 2014  
Midterm Exam Number Two  
November 12, 2014  
Answers

1. Let  $a$  and  $b$  be integers. Prove that  $x = a^2 + ab + b$  is odd iff  $a$  is odd or  $b$  is odd.

Suppose  $a$  and  $b$  are integers.

We can consider four cases.

- (a) Suppose  $a$  is even and  $b$  is even.

Then  $a = 2k$  and  $b = 2m$  for some integers  $k$  and  $m$ .

Then  $x = 4k^2 + 4km + 2m = 2(2k^2 + 2km + m)$ .

Since  $2k^2 + 2km + m$  is an integer (by Axiom 11),  $x$  is even.

- (b) Suppose  $a$  is even and  $b$  is odd.

Then  $a = 2k$  and  $b = 2m + 1$  for some integers  $k$  and  $m$ .

Then  $x = 4k^2 + 2k(2m + 1) + 2m + 1 = 2(2k^2 + k(2m + 1) + m) + 1$ .

Since  $2k^2 + k(2m + 1) + m$  is an integer (by Axiom 11),  $x$  is odd.

- (c) Suppose  $a$  is odd and  $b$  is even.

Then  $a = 2k + 1$  and  $b = 2m$  for some integers  $k$  and  $m$ .

Then

$$\begin{aligned}x &= (2k + 1)^2 + (2k + 1)(2m) + 2m \\&= 4k^2 + 4k + 1 + (2k + 1)(2m) + 2m \\&= 2(2k^2 + 2k + m(2k + 1) + m) + 1.\end{aligned}$$

Since  $2k^2 + 2k + m(2k + 1) + m$  is an integer (by Axiom 11),  $x$  is odd.

- (d) Suppose  $a$  is odd and  $b$  is odd.

Then  $a = 2k + 1$  and  $b = 2m + 1$  for some integers  $k$  and  $m$ .

Then

$$\begin{aligned}x &= (2k + 1)^2 + (2m + 1)(2k + 1) + b \\&= 4k^2 + 4k + 1 + 4mk + 2m + 2k + 1 + 2m + 1 \\&= 2(2k^2 + 2mk + 3k + 2m + 1) + 1.\end{aligned}$$

Since  $(2k^2 + 2mk + 3k + 2m + 1)$  is an integer (by Axiom 11),  $x$  is odd.

The first case shows that if  $a$  and  $b$  are even, then  $x$  is even; hence, if  $x$  is odd, then  $a$  is odd or  $b$  is odd.

The other three cases show that if  $a$  or  $b$  is odd, then  $x$  is odd.

Hence,  $x$  is odd if and only if  $a$  is odd or  $b$  is odd. ■

2. Let  $S = \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ . Define a relation  $R \subseteq S \times S$  by

$$((x_1, y_1), (x_2, y_2)) \in R \Leftrightarrow x_1 y_1 = x_2 y_2.$$

(a) Prove that  $R$  is an equivalence relation.

Suppose  $P = (x, y) \in S$ .

Since  $xy = xy$ ,  $((x, y), (x, y)) \in R$ .

That is,  $(P, P) \in R$ .

Hence,  $R$  is reflexive.

Suppose  $((x_1, y_1), (x_2, y_2)) \in R$ .

Then  $x_1 y_1 = x_2 y_2$ .

Hence,  $x_2 y_2 = x_1 y_1$ , so  $((x_2, y_2), (x_1, y_1)) \in R$ .

Thus,  $R$  is symmetric.

Suppose  $((x_1, y_1), (x_2, y_2)) \in R$  and  $((x_2, y_2), (x_3, y_3)) \in R$ .

Then  $x_1 y_1 = x_2 y_2$  and  $x_2 y_2 = x_3 y_3$ .

Hence,  $x_1 y_1 = x_3 y_3$ .

Thus,  $((x_1, y_1), (x_3, y_3)) \in R$ .

Hence,  $R$  is transitive.

Since  $R$  is reflexive, symmetric and transitive,  $R$  is an equivalence relation. ■

(b) Notice that elements of  $S$  can be viewed as points in the first quadrant of the  $xy$ -plane (i.e., the set of points  $(x, y)$  where  $x > 0$  and  $y > 0$ .) Draw a picture of one equivalence class in  $S/R$  and indicate which equivalence class it is.

Consider  $(1, 1) \in S$ .

We have  $((1, 1), (x, y)) \in R$  iff  $1 = xy$ , i.e.,  $y = 1/x$ .

Thus, the portion of the curve  $y = 1/x$  in the first quadrant is the equivalence class  $[(1, 1)]$  in  $S/R$ .

(In fact, for any positive  $j$ , the curve  $y = j/x$  represents an equivalence class in  $S/R$ .)

3. Let  $A$  and  $B$  be sets. Prove that  $A = B$  iff  $\mathcal{P}(A) = \mathcal{P}(B)$ .

Let  $A$  and  $B$  be sets.

Suppose  $A = B$ . Then  $\mathcal{P}(A) = \mathcal{P}(B)$ .

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Suppose  $x \in A$ .

Then  $\{x\} \in \mathcal{P}(A)$ , so  $\{x\} \in \mathcal{P}(B)$ .

Hence,  $\{x\} \subseteq B$ .

Since  $x \in \{x\}$ ,  $x \in B$ .

Hence, since  $x \in A$  implies  $x \in B$ ,  $A \subseteq B$ .

Now, suppose  $y \in B$ .

Then  $\{y\} \in \mathcal{P}(B)$ , so  $\{y\} \in \mathcal{P}(A)$ .

Hence,  $\{y\} \subseteq A$ .

Since  $y \in \{y\}$ ,  $y \in A$ .

Hence, since  $y \in B$  implies  $y \in A$ ,  $B \subseteq A$ .

Thus,  $A \subseteq B$  and  $B \subseteq A$ , so  $A = B$ .

Therefore,  $A = B$  iff  $\mathcal{P}(A) = \mathcal{P}(B)$ . ■

4. Prove that  $\sqrt{2} + \sqrt{3}$  is an algebraic number.

Let  $x = \sqrt{2} + \sqrt{3}$ .

Then  $(x - \sqrt{2})^2 = 3$

so  $x^2 - 2\sqrt{2}x - 1 = 0$ .

Hence,  $(-2\sqrt{2}x) = (1 - x^2)^2$ , i.e.

$8x^2 = 1 - 2x^2 + x^4$ , i.e.

$0 = x^4 - 10x^2 + 1$ .

This shows that  $x$  is a root of a polynomial with integer coefficients, and so  $x$  is an algebraic number. ■

5. Let  $\mathcal{F}$  be a family of sets, and  $B$  be a set. Prove that if  $\bigcup \mathcal{F} \subseteq B$ , then  $\mathcal{F} \subseteq \mathcal{P}(B)$ .

Let  $\mathcal{F}$  be a family of sets, and  $B$  be a set.

Suppose  $\bigcup \mathcal{F} \subseteq B$ .

Let  $x \in \mathcal{F}$ .

Let  $y \in x$ .

Then  $y \in \bigcup \mathcal{F}$ .

Hence,  $y \in B$ .

Since  $y \in x$  implies  $y \in B$ ,  $x \subseteq B$ .

Hence,  $x \in \mathcal{P}(B)$ .

Since  $x \in \mathcal{F}$  implies  $x \in \mathcal{P}(B)$ ,  $\mathcal{F} \subseteq \mathcal{P}(B)$ . ■