Math 300 D - Autumn 2014
Midterm Exam Number Two
November 12, 2014

Name: ___________________________ Student ID no.: _______________

Signature: ___________________________________________________________________

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- Complete all five questions.
- You have 50 minutes to complete the exam.
1. Let $a$ and $b$ be integers. Prove that $x = a^2 + ab + b$ is odd iff $a$ is odd or $b$ is odd.
2. Let $S = \mathbb{R}_{>0} \times \mathbb{R}_{>0}$. Define a relation $R \subseteq S \times S$ by

$$(x_1, y_1), (x_2, y_2) \in R \iff x_1 y_1 = x_2 y_2.$$ 

(a) Prove that $R$ is an equivalence relation.

(b) Notice that elements of $S \times S$ can be viewed as points in the first quadrant of the $xy$-plane (i.e., the set of points $(x, y)$ where $x > 0$ and $y > 0$.) Draw a picture of one equivalence class in $S \times S/R$ and indicate which equivalence class it is.
3. Let $A$ and $B$ be sets. Prove that $A = B$ iff $\mathcal{P}(A) = \mathcal{P}(B)$. 
4. Prove that $\sqrt{2} + \sqrt{3}$ is an algebraic number.

5. Let $\mathcal{F}$ be a family of sets, and $B$ be a set. Prove that if $\bigcup \mathcal{F} \subseteq B$, then $\mathcal{F} \subseteq \mathcal{P}(B)$. 
Axioms

Suppose $x$, $y$, and $z$ are real numbers. We will take as fact each of the following.

1. $x + y$ and $xy$ are real numbers. ($\mathbb{R}$ is closed under addition and multiplication.)

2. If $x = y$, then $x + z = y + z$ and $xz = yz$. (This is sometimes called substitution of equals.)

3. $x + y = y + x$ and $xy = yx$ (addition and multiplication are commutative in $\mathbb{R}$)

4. $(x + y) + z = x + (y + z)$ and $(xy)z = x(yz)$ (addition and multiplication are associative in $\mathbb{R}$)

5. $x(y + z) = xy + xz$ (This is the Distributive Law.)

6. $x + 0 = 0 + x = x$ and $x \cdot 1 = 1 \cdot x = x$ (0 is the additive identity; 1 is the multiplicative identity.)

7. There exists a real number $-x$ such that $x + (-x) = (-x) + x = 0$. (That is, every real number has an additive inverse in $\mathbb{R}$.)

8. If $x \neq 0$, then there exists a real number $x^{-1}$ such that $x \cdot x^{-1} = x^{-1} \cdot x = 1$. (That is, every non-zero real number has a multiplicative inverse in $\mathbb{R}$.)

9. If $x > 0$ and $y > 0$, then $x + y > 0$ and $xy > 0$.

10. Either $x > 0$, $-x > 0$, or $x = 0$.

11. If $x$ and $y$ are integers, then $-x$, $x + y$, and $xy$ are integers. (The additive inverse of an integer is an integer and $\mathbb{Z}$ is closed under addition and multiplication.)

NOTE: It is not hard to prove that $\mathbb{Q}$, the set of rational numbers is closed under addition and multiplication and that every non-zero rational number has a multiplicative inverse in $\mathbb{Q}$.

Elementary Properties of Real Numbers

The following properties of real numbers that allow us to do algebra follow from the axioms on the left.

If $x, y, z, u$, and $v$ are real numbers, then:

1. $x \cdot 0 = 0$

2. If $x + z = y + z$, then $x = y$.

3. If $x \cdot z = y \cdot z$ and $z \neq 0$, then $x = y$.

4. $-x = (-1) \cdot x$

5. $(-x) \cdot y = -(x \cdot y)$

6. $(-x) \cdot (-y) = x \cdot y$

7. If $x \cdot y = 0$, then $x = 0$ or $y = 0$.

8. If $x \leq y$ and $y \leq x$, then $x = y$.

9. If $x \leq y$ and $y \leq z$, then $x \leq z$.

10. At least one of the following is true: $x \leq y$ or $y \leq x$.

11. If $x \leq y$, then $x + z \leq y + z$.

12. If $x \leq y$ and $0 \leq z$, then $xz \leq yz$.

13. If $x \leq y$ and $z \leq 0$, then $yz \leq xz$.

14. If $x \leq y$ and $u \leq v$, then $x + u \leq y + v$.

15. If $0 \leq x \leq y$ and $0 \leq u \leq v$, then $xu \leq yv$.

16. If $x \leq y$, then $-y \leq -x$.

17. $0 \leq x^2$

18. $0 < 1$

19. If $0 < x$, then $0 < x^{-1}$.

20. If $0 < x < y$, then $0 < y^{-1} < x^{-1}$.

And here are a couple of properties of integers.

21. Every integer is either even or odd, never both.

22. The only integers that divide 1 are $-1$ and $1$. 
DeMorgan’s laws

\[ \neg(P \land Q) \text{ is equivalent to } \neg P \lor \neg Q \]
\[ \neg(P \lor Q) \text{ is equivalent to } \neg P \land \neg Q \]

Commutative Laws

\[ P \land Q \text{ is equivalent to } Q \land P \]
\[ P \lor Q \text{ is equivalent to } Q \lor P \]

Associative Laws

\[ P \land (Q \land R) \text{ is equivalent to } (P \land Q) \land R \]
\[ P \lor (Q \lor R) \text{ is equivalent to } (P \lor Q) \lor R \]

Idempotent Laws

\[ P \land P \text{ is equivalent to } P \]
\[ P \lor P \text{ is equivalent to } P \]

Distributive Laws

\[ P \land (Q \lor R) \text{ is equivalent to } (P \land Q) \lor (P \land R) \]
\[ P \lor (Q \land R) \text{ is equivalent to } (P \lor Q) \land (P \lor R) \]

Absorption Laws

\[ P \lor (P \land Q) \text{ is equivalent to } P \]
\[ P \land (P \lor Q) \text{ is equivalent to } P \]

Double Negation Law

\[ \neg\neg P \text{ is equivalent to } P \]

Conditional Laws

\[ P \rightarrow Q \text{ is equivalent to } \neg P \lor Q \]
\[ P \rightarrow Q \text{ is equivalent to } \neg(P \land \neg Q) \]

Contrapositive Laws

\[ P \rightarrow Q \text{ is equivalent to } \neg Q \rightarrow \neg P \]

Quantifier Negation Laws

\[ \neg\exists x P(x) \text{ is equivalent to } \forall x \neg P(x) \]
\[ \neg\forall x P(x) \text{ is equivalent to } \exists x \neg P(x) \]

Sets

\[ A = B \Leftrightarrow \left((x \in A) \Leftrightarrow (x \in B)\right) \]
\[ x \in A \cup B \Leftrightarrow \left((x \in A) \lor (x \in B)\right) \]
\[ x \in A \cap B \Leftrightarrow \left((x \in A) \land (x \in B)\right) \]
\[ x \in A \setminus B \Leftrightarrow (x \in A) \land (x \notin B) \]

Tautology Laws

\[ P \land (\text{a tautology}) \text{ is equivalent to } P \]
\[ P \lor (\text{a tautology}) \text{ is a tautology} \]
\[ \neg(\text{a tautology}) \text{ is a contradiction} \]

Contradiction Laws

\[ P \land (\text{a contradiction}) \text{ is a contradiction} \]
\[ P \lor (\text{a contradiction}) \text{ is equivalent to } P \]
\[ \neg(\text{a contradiction}) \text{ is a tautology} \]