

Math 300 B - Winter 2015  
Final Exam  
March 18, 2015

Name: \_\_\_\_\_

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Signature: \_\_\_\_\_

|       |    |  |
|-------|----|--|
| 1     | 10 |  |
| 2     | 10 |  |
| 3     | 10 |  |
| 4     | 10 |  |
| 5     | 10 |  |
| 6     | 10 |  |
| Total | 60 |  |

- Complete all 6 questions.
- You have 110 minutes to complete the exam.

1. Let  $a$  and  $b$  be integers. Prove that  $3|a^2 + b^2$  if and only if  $3|a$  and  $3|b$ .

2. Let  $A$  and  $B$  be sets. Prove that

$$A \subseteq B$$

if and only if

$$\mathcal{P}(A) \subseteq \mathcal{P}(B).$$

3. Use induction to prove that

$$\sum_{i=1}^n \frac{1}{i(i+1)} = \frac{n}{n+1}$$

for all  $n \in \mathbb{Z}_{>0}$ .

4. Suppose  $A, B$  and  $C$  are sets with  $A \subseteq C$  and  $B \subseteq C$ . Suppose  $A \cap B = \emptyset$ , and that  $A \sim B$ . Prove that

$$C \setminus A \sim C \setminus B.$$

5. Suppose  $A$ ,  $B$  and  $C$  are sets. Suppose  $f : A \rightarrow B$  and  $g : B \rightarrow C$ .

(a) Prove that if  $f$  is onto and  $g$  is not one-to-one, then  $g \circ f$  is not one-to-one.

(b) Prove that if  $f$  is not onto and  $g$  is one-to-one, then  $g \circ f$  is not onto.

6. Let  $A = \mathbb{R} \times \mathbb{R} \setminus \{(0, 0)\}$ .

Thus,  $A$  is the  $xy$ -plane without the origin.

Define a relation  $R$  on  $A$  by

$((x_1, y_1), (x_2, y_2)) \in R \Leftrightarrow (x_1, y_1)$  and  $(x_2, y_2)$  lie on a line which passes through the origin.

Prove that  $R$  is an equivalence relation.

## Set notation

$$A = B \Leftrightarrow ((x \in A) \Leftrightarrow (x \in B))$$

$$A \subseteq B \Leftrightarrow ((x \in A) \rightarrow (x \in B))$$

$$x \in A \cup B \Leftrightarrow ((x \in A) \vee (x \in B))$$

$$x \in A \cap B \Leftrightarrow ((x \in A) \wedge (x \in B))$$

$$x \in A \setminus B \Leftrightarrow (x \in A) \wedge (x \notin B)$$

$$A \sim B \Leftrightarrow \text{there exists a bijection } f : A \rightarrow B$$

## Axioms

Suppose  $x, y$ , and  $z$  are real numbers. We will take as fact each of the following.

1.  $x + y$  and  $xy$  are real numbers. ( $\mathbb{R}$  is *closed* under addition and multiplication.)
2. If  $x = y$ , then  $x + z = y + z$  and  $xz = yz$ . (This is sometimes called *substitution of equals*.)
3.  $x + y = y + x$  and  $xy = yx$  (addition and multiplication are *commutative* in  $\mathbb{R}$ )
4.  $(x + y) + z = x + (y + z)$  and  $(xy)z = x(yz)$  (addition and multiplication are *associative* in  $\mathbb{R}$ )
5.  $x(y + z) = xy + xz$  (This is the *Distributive Law*.)
6.  $x + 0 = 0 + x = x$  and  $x \cdot 1 = 1 \cdot x = x$  (0 is the *additive identity*; 1 is the *multiplicative identity*.)
7. There exists a real number  $-x$  such that  $x + (-x) = (-x) + x = 0$ . (That is, every real number has an *additive inverse* in  $\mathbb{R}$ .)
8. If  $x \neq 0$ , then there exists a real number  $x^{-1}$  such that  $x \cdot x^{-1} = x^{-1} \cdot x = 1$ . (That is, every non-zero real number has a *multiplicative inverse* in  $\mathbb{R}$ .)
9. If  $x > 0$  and  $y > 0$ , then  $x + y > 0$  and  $xy > 0$ .
10. Either  $x > 0$ ,  $-x > 0$ , or  $x = 0$ .
11. If  $x$  and  $y$  are integers, then  $-x$ ,  $x + y$ , and  $xy$  are integers. (The additive inverse of an integer is an integer and  $\mathbb{Z}$  is closed under addition and multiplication.)

NOTE: It is not hard to prove that  $\mathbb{Q}$ , the set of rational numbers is closed under addition and multiplication and that every non-zero rational number has a multiplicative inverse in  $\mathbb{Q}$ .

## Elementary Properties of Real Numbers

If  $x, y, z, u$ , and  $v$  are real numbers, then:

1.  $x \cdot 0 = 0$
  2. If  $x + z = y + z$ , then  $x = y$ .
  3. If  $x \cdot z = y \cdot z$  and  $z \neq 0$ , then  $x = y$ .
  4.  $-x = (-1) \cdot x$
  5.  $(-x) \cdot y = -(x \cdot y)$
  6.  $(-x) \cdot (-y) = x \cdot y$
  7. If  $x \cdot y = 0$ , then  $x = 0$  or  $y = 0$ .
  8. If  $x \leq y$  and  $y \leq x$ , then  $x = y$ .
  9. If  $x \leq y$  and  $y \leq z$ , then  $x \leq z$ .
  10. At least one of the following is true:  $x \leq y$  or  $y \leq x$ .
  11. If  $x \leq y$ , then  $x + z \leq y + z$ .
  12. If  $x \leq y$  and  $0 \leq z$ , then  $xz \leq yz$ .
  13. If  $x \leq y$  and  $z \leq 0$ , then  $yz \leq xz$ .
  14. If  $x \leq y$  and  $u \leq v$ , then  $x + u \leq y + v$ .
  15. If  $0 \leq x \leq y$  and  $0 \leq u \leq v$ , then  $xu \leq yv$ .
  16. If  $x \leq y$ , then  $-y \leq -x$ .
  17.  $0 \leq x^2$
  18.  $0 < 1$
  19. If  $0 < x$ , then  $0 < x^{-1}$ .
  20. If  $0 < x < y$ , then  $0 < y^{-1} < x^{-1}$ .
- And here are a couple of properties of integers.
21. Every integer is either even or odd, never both.
  22. The only integers that divide 1 are  $-1$  and  $1$ .