1. Let $A$ and $B$ be disjoint sets. Suppose there is a bijection from $A$ to $I_n$ and there is a bijection from $B$ to $I_m$. Prove that there exists a bijection from $A \cup B$ to $I_{m+n}$.

**Proof:** Let $A$ and $B$ be disjoint sets. Suppose there is a bijection from $A$ to $I_n$ and there is a bijection from $B$ to $I_m$. Say $f : A \rightarrow I_n$ and $g : B \rightarrow I_m$ are bijections.

Define $h : A \cup B \rightarrow I_{m+n}$ by

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) + n & \text{if } x \in B. \end{cases}$$

Suppose $h(x_1) = h(x_2)$ for some $x_1$ and $x_2 \in A \cup B$.

Suppose $x_1$ and $x_2$ are not both in $A$. Without loss of generality, suppose $x_1 \in A$ and $x_2 \in B$. Then $h(x_1) \leq n$ and $h(x_2) \geq n + 1$, so $h(x_1) \neq h(x_2)$. This is a contradiction, so either $x_1$ and $x_2$ are both in $A$, or they are both in $B$.

Suppose $x_1, x_2 \in A$. Then $f(x_1) = f(x_2)$, so $x_1 = x_2$.

Suppose $x_1, x_2 \in B$. Then $g(x_1) + n = g(x_2) + n$, so $g(x_1) = g(x_2)$, and hence $x_1 = x_2$.

Thus, $x_1 = x_2$, so $h$ is one-to-one.

Suppose $y \in I_{m+n}$.

Suppose $1 \leq y \leq n$. Then $y \in I_n$, and since $f$ is bijective, there is a $\hat{x} \in A$ such that $f(\hat{x}) = y$ and $h(\hat{x}) = y$.

Suppose $y > n$. Then $y - n \in I_m$, and since $g$ is bijective, there is a $\hat{x} \in B$ such that $g(\hat{x}) = y - n$ and $h(\hat{x}) = y$.

Thus, there is a $\hat{x} \in A \cup B$ such that $h(\hat{x}) = y$ and so $h$ is surjective.

Therefore, $h$ is a bijection. ■

2. Use induction to prove that, for all integers $n \geq 0$,

$$8|5^n + 12n - 1.$$ 

**Proof:** Let $P(n) = "8|5^n + 12n - 1"$.

If $n = 0$, then $5^0 + 12 \cdot 0 - 1 = 0$ and $8|0$, so $P(0)$ is true.

Suppose $P(k)$ is true for some $k \geq 0$.

Then $8|5^k + 12k - 1$, so there is an $m \in \mathbb{Z}$ with $5^k + 12k - 1 = 8m$. 

Let $n = k + 1$.

Then $5^n + 12n - 1 = 5^{k+1} + 12(k+1) - 1 = 5 \cdot 5^k + 12k + 11$.

Since $5^k + 12k - 1 = 8m$, then

$$5 \cdot (5^k + 12k - 1) + 11 = 5 \cdot 8m + 11 = 40m + 11.$$ 

Since $40m + 11$ is divisible by 8, then $5^n + 12n - 1$ is divisible by 8.

Therefore, $P(n)$ is true for all $n \geq 0$. 

$\blacksquare$
3. (a) Let \( A \) be the set of all real functions \( f : \mathbb{R} \to \mathbb{R} \). Define a relation \( R \) on \( A \) by:

\[
(f, g) \in R \iff \text{there exists a real constant } k \text{ such that } f(x) = g(x) + k \text{ for all } x \in \mathbb{R}.
\]

Prove that \( R \) is an equivalence relation.

**Proof:** Let \( f : \mathbb{R} \to \mathbb{R} \). Since \( f(x) = f(x) + 0 \) for all \( x \in \mathbb{R} \), \((f, f) \in R \).

Hence, \( R \) is reflexive.

Suppose \( (f, g) \in R \).

Then there exists \( k \in \mathbb{R} \) with \( f(x) = g(x) + k \) for all \( x \in \mathbb{R} \).

Then \( g(x) = f(x) + (-k) \) for all \( x \in \mathbb{R} \). Since \(-k \in \mathbb{R} \), we conclude that \((g, f) \in R \).

Hence, \( R \) is symmetric.

Suppose \( (f, g) \in R \) and \((g, h) \in R \).

Then there exist \( k_1, k_2 \in \mathbb{R} \) such that \( f(x) = g(x) + k_1 \) and \( g(x) = f(x) + k_2 \) for all \( x \in \mathbb{R} \).

Then \( f(x) = h(x) + k_1 + k_2 \) for all \( x \in \mathbb{R} \) and \( k_1 + k_2 \in \mathbb{R} \) so \((f, h) \in R \).

Hence, \( R \) is transitive, and so \( R \) is an equivalence relation. \( \blacksquare \)

(b) Define a relation \( R \) on \( \mathbb{R} \) by:

\[
(x, y) \in R \iff |x - y| < 1
\]

Prove that \( R \) is not an equivalence relation.

**Proof:**

\((2, 2.5) \in R \) since \( |2 - 2.5| = 0.5 < 1 \).

\((2.5, 3) \in R \) since \( |2.5 - 3| = 0.5 < 1 \).

However, \((2, 3) \notin R \) since \( |2 - 3| = 1 \notin 1 \).

So \( R \) is not transitive, and hence \( R \) is not an equivalence relation. \( \blacksquare \)

4. Let \( A \) and \( B \) be sets. Prove that \( \mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B) \).

**Proof:** Suppose \( x \in \mathcal{P}(A \cap B) \). Then \( x \subseteq A \cap B \).

Suppose \( z \in x \). Then \( z \in A \cap B \), so \( z \in A \) and \( z \in B \).

Hence, \( x \subseteq A \) and \( x \subseteq B \), i.e., \( x \in \mathcal{P}(A) \cap \mathcal{P}(B) \).

Thus, \( \mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B) \).

Now, suppose \( x \in \mathcal{P}(A) \cap \mathcal{P}(B) \). Then \( x \in \mathcal{P}(A) \) and \( x \in \mathcal{P}(B) \), i.e., \( x \subseteq A \) and \( x \subseteq B \).
Suppose \( y \in x \). Then \( y \in A \) and \( y \in B \), so \( y \in A \cap B \).

Hence, \( x \subseteq A \cap B \), i.e., \( x \in \mathcal{P}(A \cap B) \).

Thus \( \mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B) \).

Therefore, \( \mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B) \). ■

5. (a) Let \( m \in \mathbb{Z} \) and suppose \( m > 1 \). Suppose \( a, b, c \in \mathbb{Z} \).

Prove that if \( a \equiv b \pmod{m} \), then \( ac \equiv bc \pmod{m} \).

**Proof:** Suppose \( a, b, c \in \mathbb{Z} \). Suppose \( a \equiv b \pmod{m} \).

Then there exists a \( k \in \mathbb{Z} \) such that \( a - b = mk \).

Then \( ac - bc = mkc \) and hence \( m|ac - bc \), i.e., \( ac \equiv bc \pmod{m} \). ■.

(b) Prove that if \( n \) is an integer then \( n^2 \equiv 0, 1, \text{ or } 4 \pmod{8} \).

**Proof:**

Let \( n \) be an integer. Then \( n \equiv 0, 1, 2, 3, 4, 5, 6, \text{ or } 7 \pmod{8} \). And so

<table>
<thead>
<tr>
<th>( n \mod 8 )</th>
<th>( n^2 \mod 8 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<tr>
<td>1</td>
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<td>4</td>
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<td>6</td>
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<td>7</td>
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</table>

Thus, this direct calculation shows that \( n^2 \equiv 0, 1, \text{ or } 4 \pmod{8} \). ■

6. Let \( A, B \) and \( C \) be sets. Let \( f : A \to B \), and \( g : B \to C \).

(a) Suppose \( g \circ f : A \to C \) is one-to-one. Is \( f \) necessarily one-to-one? Prove your answer.

\( f \) is necessarily one-to-one.

**Proof:** Suppose \( f \) is not one-to-one.

Then there exist \( a_1, a_2 \in A \), \( a_1 \neq a_2 \), with \( f(a_1) = f(a_2) \).

Then \( g(f(a_1)) = g(f(a_2)) \). But \( a_1 \neq a_2 \), so \( g \circ f \) is not one-to-one. This is a contradiction.

Hence \( f \) is one-to-one. ■

(b) Suppose \( g \circ f : A \to C \) is one-to-one. Is \( g \) necessarily one-to-one? Prove your answer.

\( g \) is not necessarily one-to-one.

**Proof:** We may defined \( A = \{a\} \), \( B = \{b_1, b_2\} \), and \( C = \{c\} \). Then define \( f = \{(a, b_2)\} \), \( g = \{(b_1, c), (b_2, c)\} \).

Then \( g \circ f = \{(a, c)\} \), and \( g \circ f \) is one-to-one though \( g \) is not.

Alternatively, define \( A = \mathbb{Z}_{>0} \), \( B = \mathbb{Z} \), and \( C = \mathbb{Z} \).

Let \( f(x) = x \) and \( g(x) = |x| \). Then \( (g \circ f)(x) = |x| \) is one-to-one from \( \mathbb{Z}_{>0} \) to \( \mathbb{Z} \), but \( g \) is not one-to-one from \( \mathbb{Z} \) to \( \mathbb{Z} \). ■