

Math 300 D - Winter 2014
Final Exam
March 18, 2014
Solutions

1. Let A and B be disjoint sets. Suppose there is a bijection from A to I_n and there is a bijection from B to I_m . Prove that there exists a bijection from $A \cup B$ to I_{m+n} .

Proof: Let A and B be disjoint sets. Suppose there is a bijection from A to I_n and there is a bijection from B to I_m . Say $f : A \rightarrow I_n$ and $g : B \rightarrow I_m$ are bijections.

Define $h : A \cup B \rightarrow I_{m+n}$ by

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) + n & \text{if } x \in B. \end{cases}$$

Suppose $h(x_1) = h(x_2)$ for some x_1 and $x_2 \in A \cup B$.

Suppose x_1 and x_2 are not both in A . Without loss of generality, suppose $x_1 \in A$ and $x_2 \in B$. Then $h(x_1) \leq n$ and $h(x_2) \geq n + 1$, so $h(x_1) \neq h(x_2)$. This is a contradiction, so either x_1 and x_2 are both in A , or they are both in B .

Suppose $x_1, x_2 \in A$. Then $f(x_1) = f(x_2)$, so $x_1 = x_2$.

Suppose $x_1, x_2 \in B$. Then $g(x_1) + n = g(x_2) + n$, so $g(x_1) = g(x_2)$, and hence $x_1 = x_2$.

Thus, $x_1 = x_2$, so h is one-to-one.

Suppose $y \in I_{m+n}$.

Suppose $1 \leq y \leq n$. Then $y \in I_n$, and since f is bijective, there is a $\hat{x} \in A$ such that $f(\hat{x}) = y$ and $h(\hat{x}) = y$.

Suppose $y > n$. Then $y - n \in I_m$, and since g is bijective, there is a $\hat{x} \in B$ such that $g(\hat{x}) = y - n$ and $h(\hat{x}) = y$.

Thus, there is a $\hat{x} \in A \cup B$ such that $h(\hat{x}) = y$ and so h is surjective.

Therefore, h is a bijection. ■

2. Use induction to prove that, for all integers $n \geq 0$,

$$8 \mid 5^n + 12n - 1.$$

Proof: Let $P(n) = "8 \mid 5^n + 12n - 1"$.

If $n = 0$, then $5^0 + 12 \cdot 0 - 1 = 0$ and $8 \mid 0$, so $P(0)$ is true.

Suppose $P(k)$ is true for some $k \geq 0$.

Then $8 \mid 5^k + 12k - 1$, so there is an $m \in \mathbb{Z}$ with $5^k + 12k - 1 = 8m$.

Then

$$\begin{aligned}5^{k+1} + 12(k+1) - 1 &= 5(5^k + 12k - 1) - 60k + 5 + 12(k+1) - 1 \\&= 5(5^k + 12k - 1) - 48k + 16 \\&= 5 \cdot 8m + 8(-6k + 2) \\&= 8(5m - 6k + 2).\end{aligned}$$

So, $8 \mid 5^{k+1} + 12(k+1) - 1$.

Hence, $P(k)$ implies $P(k+1)$, and $P(0)$ is true, so by induction, $P(k)$ is true for all $k \geq 0$. ■

3. (a) Let A be the set of all real functions $f : \mathbb{R} \rightarrow \mathbb{R}$. Define a relation R on A by:

$$(f, g) \in R \Leftrightarrow \text{there exists a real constant } k \text{ such that } f(x) = g(x) + k \text{ for all } x \in \mathbb{R}.$$

Prove that R is an equivalence relation.

Proof: Let $f : \mathbb{R} \rightarrow \mathbb{R}$. Since $f(x) = f(x) + 0$ for all $x \in \mathbb{R}$, $(f, f) \in R$.

Hence, R is reflexive.

Suppose $(f, g) \in R$.

Then there exists $k \in \mathbb{R}$ with $f(x) = g(x) + k$ for all $x \in \mathbb{R}$.

Then $g(x) = f(x) + (-k)$ for all $x \in \mathbb{R}$. Since $-k \in \mathbb{R}$, we conclude that $(g, f) \in R$.

Hence, R is symmetric.

Suppose $(f, g) \in R$ and $(g, h) \in R$.

Then there exist $k_1, k_2 \in \mathbb{R}$ such that $f(x) = g(x) + k_1$ and $g(x) = h(x) + k_2$ for all $x \in \mathbb{R}$.

Then $f(x) = h(x) + k_1 + k_2$ for all $x \in \mathbb{R}$ and $k_1 + k_2 \in \mathbb{R}$ so $(f, h) \in R$.

Hence, R is transitive, and so R is an equivalence relation. ■

- (b) Define a relation R on $\mathbb{R}$ by:

$$(x, y) \in R \Leftrightarrow |x - y| < 1$$

Prove that R is not an equivalence relation.

Proof:

$(2, 2.5) \in R$ since $|2 - 2.5| = 0.5 < 1$.

$(2.5, 3) \in R$ since $|2.5 - 3| = 0.5 < 1$.

However, $(2, 3) \notin R$ since $|2 - 3| = 1 \not< 1$.

So R is not transitive, and hence R is not an equivalence relation. ■

4. Let A and B be sets. Prove that $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.

Proof: Suppose $x \in \mathcal{P}(A \cap B)$. Then $x \subseteq A \cap B$.

Suppose $z \in x$. Then $z \in A \cap B$, so $z \in A$ and $z \in B$.

Hence, $x \subseteq A$ and $x \subseteq B$, i.e., $x \in \mathcal{P}(A) \cap \mathcal{P}(B)$.

Thus, $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$.

Now, suppose $x \in \mathcal{P}(A) \cap \mathcal{P}(B)$. Then $x \in \mathcal{P}(A)$ and $x \in \mathcal{P}(B)$, i.e., $x \subseteq A$ and $x \subseteq B$.

Suppose $y \in x$. Then $y \in A$ and $y \in B$, so $y \in A \cap B$.

Hence, $x \subseteq A \cap B$, i.e., $x \in \mathcal{P}(A \cap B)$.

Thus $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$.

Therefore, $\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$. ■

5. (a) Let $m \in \mathbb{Z}$ and suppose $m > 1$. Suppose $a, b, c \in \mathbb{Z}$.

Prove that if $a \equiv b \pmod{m}$, then $ac \equiv bc \pmod{m}$.

Proof: Suppose $a, b, c \in \mathbb{Z}$. Suppose $a \equiv b \pmod{m}$.

Then there exists a $k \in \mathbb{Z}$ such that $a - b = mk$.

Then $ac - bc = mkc$ and hence $m \mid ac - bc$, i.e., $ac \equiv bc \pmod{m}$. ■

- (b) Prove that if n is an integer then $n^2 \equiv 0, 1, \text{ or } 4 \pmod{8}$.

Proof:

Let n be an integer. Then $n \equiv 0, 1, 2, 3, 4, 5, 6, \text{ or } 7 \pmod{8}$. And so

$n \pmod{8}$	$n^2 \pmod{8}$
0	0
1	1
2	4
3	1
4	0
5	1
6	4
7	1

Thus, this direct calculation shows that $n^2 \equiv 0, 1, \text{ or } 4 \pmod{8}$. ■

6. Let A, B and C be sets. Let $f : A \rightarrow B$, and $g : B \rightarrow C$.

- (a) Suppose $g \circ f : A \rightarrow C$ is one-to-one. Is f necessarily one-to-one? Prove your answer.

f is necessarily one-to-one.

Proof: Suppose f is not one-to-one.

Then there exist $a_1, a_2 \in A$, $a_1 \neq a_2$, with $f(a_1) = f(a_2)$.

Then $g(f(a_1)) = g(f(a_2))$. But $a_1 \neq a_2$, so $g \circ f$ is not one-to-one. This is a contradiction.

Hence f is one-to-one. ■

- (b) Suppose $g \circ f : A \rightarrow C$ is one-to-one. Is g necessarily one-to-one? Prove your answer.

g is not necessarily one-to-one.

Proof: We may define $A = \{a\}$, $B = \{b_1, b_2\}$, and $C = \{c\}$. Then define $f = \{(a, b_1)\}$, $g = \{(b_1, c), (b_2, c)\}$.

Then $g \circ f = \{(a, c)\}$, and $g \circ f$ is one-to-one though g is not.

Alternatively, define $A = \mathbb{Z}_{>0}$, $B = \mathbb{Z}$, and $C = \mathbb{Z}$.

Let $f(x) = x$ and $g(x) = |x|$. Then $(g \circ f)(x) = |x|$ is one-to-one from $\mathbb{Z}_{>0}$ to $\mathbb{Z}$, but g is not one-to-one from $\mathbb{Z}$ to $\mathbb{Z}$. ■