

Math 300 A - Winter 2013
Final Exam
March 20, 2013
Answers

1. Let A, B and C be sets. Prove that $(A \cap B) \setminus C = (A \setminus C) \cap (B \setminus C)$.

We have:

$$\begin{aligned}x &\in (A \cap B) \setminus C \\ \Leftrightarrow x &\in (A \cap B) \wedge x \notin C \\ \Leftrightarrow (x &\in A \wedge x \in B) \wedge x \notin C \\ \Leftrightarrow (x &\in A \wedge x \notin C) \wedge (x \in B \wedge x \notin C) \\ \Leftrightarrow (x &\in A \setminus C) \wedge (x \in B \setminus C) \\ \Leftrightarrow x &\in ((A \setminus C) \cap (B \setminus C))\end{aligned}$$

Hence, $(A \cap B) \setminus C = (A \setminus C) \cap (B \setminus C)$.

2. Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets.

Prove that if $\cup \mathcal{F} \not\subseteq \cup \mathcal{G}$, then there is some $S \in \mathcal{F}$ such that for all $T \in \mathcal{G}$, $S \not\subseteq T$.

We prove the contrapositive. Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets.

Suppose that for all $s \in \mathcal{F}$, there exists a $T \in \mathcal{G}$ such that $S \subset T$.

Suppose $a \in \cup \mathcal{F}$.

Then there exists a set $S \in \mathcal{F}$ such that $a \in S$.

Then, by our earlier assumption, there exists a $T \in \mathcal{G}$ such that $S \subset T$.

Since $a \in S$, $a \in T$.

Since $T \in \mathcal{G}$, $a \in \cup \mathcal{G}$.

Hence, $\cup \mathcal{F} \subset \cup \mathcal{G}$, and the theorem is proved.

3. Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$. Fill in the blank and prove the statement:

$f \cap g$ is a function from $\mathbb{R}$ to $\mathbb{R}$ iff _____ .

The correct statement is

$f \cap g$ is a function from $\mathbb{R}$ to $\mathbb{R}$ iff $f = g$.

Suppose $f = g$. Then $f \cap g = f$ so $f \cap g : \mathbb{R} \rightarrow \mathbb{R}$.

Now, suppose $f \neq g$.

Then $\exists (a, b) \in f$ such that $(a, b) \notin g$, or $\exists (a, b) \in g$ such that $(a, b) \notin f$.

Without loss of generality, suppose $\exists(a, b) \in f$ such that $(a, b) \notin g$.

Since $f : \mathbb{R} \rightarrow \mathbb{R}$, b is unique: if $(a, x) \in f$, then $x = b$.

Since $g : \mathbb{R} \rightarrow \mathbb{R}$, there exists a unique c in $\mathbb{R}$ such that $(a, c) \in g$.

Since $(a, b) \notin g$, $c \neq b$.

Suppose $\exists k \in \mathbb{R}$ such that $(a, k) \in f \cap g$.

Then $(a, k) \in g$, so $k = c$. But $(a, k) \in f$, so $k = b$. Since $b \neq c$, this is a contradiction.

Hence, there does not exist a $k \in \mathbb{R}$ such that $(a, k) \in f \cap g$.

So, $f \cap g$ is not a function from $\mathbb{R}$ to $\mathbb{R}$.

Therefore, $f \cap g$ is a function from $\mathbb{R}$ to $\mathbb{R}$ iff $f = g$.

4. Show that $\mathbb{Z}_{>0}$ is equinumerous with $\mathbb{Z}$ by giving an explicit example of a bijection from $\mathbb{Z}_{>0}$ to $\mathbb{Z}$. Prove that your function is a bijection.

The idea for the bijection is to map 1 to 0, 2 to 1, 3 to -1, 4 to 2, 5 to -2, etc. That is, let $f : \mathbb{Z}_{>0} \rightarrow \mathbb{Z}$ be defined by

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ -\frac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

We now show that f is a bijection.

First, note that if n is even, then $f(n) > 0$. If n is odd, then $f(n) \leq 0$. Hence, $f(n) > 0$ iff n is even.

Now, suppose n_1 and $n_2 \in \mathbb{Z}$ and $f(n_1) = f(n_2)$.

Suppose $f(n_1) > 0$.

Then $f(n_2) > 0$.

Hence n_1 and n_2 are even, and so

$$\frac{n_1}{2} = \frac{n_2}{2}.$$

Thus $n_1 = n_2$.

Now, suppose $f(n_1) \leq 0$. Then $f(n_2) \leq 0$, and so n_1 and n_2 are odd.

Hence,

$$\begin{aligned} -\frac{n_1 - 1}{2} &= -\frac{n_2 - 1}{2} \\ n_1 - 1 &= n_2 - 1 \\ n_1 &= n_2. \end{aligned}$$

Thus, f is injective.

To show f is surjection, suppose $m \in \mathbb{Z}$.

Suppose $m > 0$. Then $2m \in \mathbb{Z}_{>0}$, $2m$ is even, and $f(2m) = m$.

Suppose $m < 0$. Then $1 - 2m \in \mathbb{Z}_{>0}$, $1 - 2m$ is odd, and $f(1 - 2m) = m$.

Thus, f is surjective.

Therefore, f is a bijection.

5. Use induction to prove that, for $n \in \mathbb{Z}_{>0}$,

$$\sum_{i=1}^n \frac{1}{i(i+1)} = \frac{n}{n+1}.$$

Let $n = 1$. Then

$$\sum_{i=1}^n \frac{1}{i(i+1)} = \sum_{i=1}^1 \frac{1}{i(i+1)} = \frac{1}{1(1+1)} = \frac{1}{2} = \frac{1}{n+1}$$

so the statement is true for $n = 1$.

Suppose the statement is true for a positive integer n .

Then

$$\begin{aligned} \sum_{i=1}^{n+1} \frac{1}{i(i+1)} &= \sum_{i=1}^n \frac{1}{i(i+1)} + \frac{1}{(n+1)(n+2)} \\ &= \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} \\ &= \frac{n(n+2) + 1}{(n+1)(n+2)} \\ &= \frac{n^2 + 2n + 1}{(n+1)(n+2)} \\ &= \frac{(n+1)^2}{(n+1)(n+2)} \\ &= \frac{n+1}{n+2}. \end{aligned}$$

Thus the statement is true for all positive integers, n .

6. Let A be the set of all functions from $\mathbb{R}$ to $\mathbb{R}$.

Let $M = \{(f, g) \in A \times A : \exists \lambda \in \mathbb{Q}, \lambda \neq 0, \text{ such that } g(x) = \lambda f(x) \forall x \in \mathbb{R}\}$.

Prove that M is an equivalence relation.

Let $f \in A$. Then $f(x) = (1)f(x) \forall x \in \mathbb{R}$ and $1 \in \mathbb{Q}$.

Hence, $(f, f) \in M$. So, M is reflexive.

Suppose $(f, g) \in M$.

Then $\exists \lambda \in \mathbb{Q}, \lambda \neq 0$ such that $g(x) = \lambda f(x) \forall x \in \mathbb{R}$.

Then $f(x) = \frac{1}{\lambda} g(x) \forall x \in \mathbb{R}$, and since $\lambda \in \mathbb{Q}, \frac{1}{\lambda} \in \mathbb{Q}$.

Hence, $(g, f) \in M$. So, M is symmetric.

Suppose (f, g) and $(g, h) \in M$.

Then $\exists \lambda_1 \in \mathbb{Q}$ such that $f(x) = \lambda_1 g(x) \forall x \in \mathbb{R}$ and $\exists \lambda_2 \in \mathbb{Q}$ such that $g(x) = \lambda_2 h(x) \forall x \in \mathbb{R}$.

Then $f(x) = \lambda_1 \lambda_2 h(x) \forall x \in \mathbb{R}$.

Since $\lambda_1 \in \mathbb{Q}$ and $\lambda_2 \in \mathbb{Q}$, $\lambda_1 \lambda_2 \in \mathbb{Q}$.

Hence, $(f, h) \in M$. So, M is transitive.

Thus, M is an equivalence relation.

7. Let R be the equivalence relation on $\mathbb{R}$ defined by:

$$\text{For } a \text{ and } b \text{ in } \mathbb{R}, (a, b) \in R \Leftrightarrow a - b \in \mathbb{Q}.$$

Prove $\mathbb{Q} \in \mathbb{R}/R$.

$\mathbb{R}/R$ is the set of equivalence classes of $\mathbb{R}$ under equivalence relation R .

Consider the class $[0]$.

Let $x \in [0]$.

Then $x - 0 \in \mathbb{Q}$, so $x \in \mathbb{Q}$.

Hence, $[0] \subseteq \mathbb{Q}$.

Now, let $y \in \mathbb{Q}$.

Then $y - 0 \in \mathbb{Q}$, so $y \in [0]$.

Hence, $\mathbb{Q} \subseteq [0]$.

Thus, $\mathbb{Q} = [0] \in \mathbb{R}/R$.