

Math 300 B - Spring 2025
Final Exam
June 11, 2025
Solutions

1. *Prove that* $11 \mid 2^{209} + 3^{1723}$.

We have

$$2^2 \equiv 4 \pmod{11}$$

$$2^3 \equiv 8 \pmod{11}$$

$$2^4 \equiv 5 \pmod{11}$$

$$2^5 \equiv -1 \pmod{11}$$

so

$$2^{10} \equiv (2^5)^2 \equiv 1 \pmod{11}.$$

Hence,

$$2^{209} = 2^{200}2^9 = (2^{10})^{20}2^9 \equiv 2^9 \equiv (2^3)^3 \equiv (-3)^3 \equiv -5 \pmod{11}$$

Also,

$$3^2 \equiv -2 \pmod{11}$$

$$3^3 \equiv 5 \pmod{11}$$

$$3^4 \equiv 4 \pmod{11}$$

$$3^5 \equiv 1 \pmod{11}.$$

Hence,

$$3^{1723} = 3^{1720+3} = 3^{344 \cdot 5 + 3} = (3^5)^{344}3^3 \equiv 5 \pmod{11}.$$

Finally, we have

$$2^{209} + 3^{1723} \equiv (-5) + 5 \equiv 0 \pmod{11}$$

and therefore

$$11 \mid 2^{209} + 3^{1723}. \quad \blacksquare$$

2. Let A, B and C be sets. Prove that

$$A \cap (B \cup C) \subseteq B \cap C$$

if and only if

$$A \cap B = A \cap C.$$

Let A, B and C be sets.

Suppose $A \cap (B \cup C) \subseteq B \cap C$.

Suppose $x \in A \cap B$.

Then $x \in A$ and $x \in B$.

So $x \in B \cup C$, and so $x \in A \cap (B \cup C)$.

Hence $x \in B \cap C$ so $x \in C$.

Thus $x \in A \cap C$.

Thus, $x \in A \cap B$ implies $x \in A \cap C$, so $A \cap B \subseteq A \cap C$.

Now, suppose $x \in A \cap C$.

Then $x \in A$ and $x \in C$.

So $x \in B \cup C$, and so $x \in A \cap (B \cup C)$.

Hence $x \in B \cap C$ so $x \in B$.

Thus, $x \in A \cap B$.

Thus, $x \in A \cap C$ implies $x \in A \cap B$, so $A \cap C \subseteq A \cap B$.

Hence, $A \cap B = A \cap C$.

Now, suppose $A \cap B = A \cap C$.

Suppose $x \in A \cap (B \cup C)$.

Then $x \in A$.

Also, $x \in B \cup C$, so $x \in B$ or $x \in C$.

Suppose $x \in B$. Then $x \in A \cap B$, so $x \in A \cap C$.

Hence, $x \in C$ so $x \in B \cap C$.

Suppose $x \in C$. Then $x \in A \cap C$, so $x \in A \cap B$.

Hence, $x \in B$ so $x \in B \cap C$.

Thus, $x \in A \cap (B \cup C)$ implies $x \in B \cap C$, so $A \cap (B \cup C) \subseteq B \cap C$.

Hence, $A \cap (B \cup C) \subseteq B \cap C$ if and only if $A \cap B = A \cap C$. ■

3. (a) Let $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2\}$, and $C = \{c_1, c_2, c_3\}$.
Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be defined by

$$f = \{(a_1, b_1), (a_2, b_1), (a_3, b_2)\} \text{ and } g = \{(b_1, c_1), (b_2, c_2)\}.$$

List all elements of the set $g \circ f$.

$$\{(a_1, c_1), (a_2, c_1), (a_3, c_2)\}.$$

- (b) Let $A = [1, 5] \subseteq \mathbb{R}$ and $B = [-2, 6] \subseteq \mathbb{R}$.

Show that $f : A \rightarrow B$ given by $f(x) = 2x - 4$ is a bijection.

Suppose $f(x_1) = f(x_2)$ for some $x_1, x_2 \in A$.

Then $2x_1 - 4 = 2x_2 - 4$ so $2(x_1 - x_2) = 0$ and hence $x_1 = x_2$.

Thus, f is one-to-one.

Suppose $y \in B$.

Let $x = \frac{y+4}{2}$.

Since $-2 \leq y \leq 6$, $2 \leq y + 4 \leq 10$ and so $1 \leq x \leq 5$.

Thus, $x \in A$ and we have

$$f(x) = f\left(\frac{y+4}{2}\right) = y.$$

Hence, for all $y \in B$, $\exists x \in A$ with $f(x) = y$, so f is onto.

Since f is one-to-one and onto, f is a bijection. ■

4. Use induction to prove that

$$\sum_{n=1}^k n(n+1) = \frac{1}{3}k(k+1)(k+2)$$

for all $k \in \mathbb{Z}_{>0}$.

For $k \in \mathbb{Z}_{>0}$, let

$$P(k) = \text{"} \sum_{n=1}^k n(n+1) = \frac{1}{3}k(k+1)(k+2) \text{"}.$$

Let $k = 1$.

Then

$$\sum_{n=1}^k n(n+1) = 1(2) = 2 = \frac{1}{3}(1)(2)(3) = \frac{1}{3}k(k+1)(k+2).$$

So $P(1)$ is true.

Now, suppose $P(m)$ is true from some $m \in \mathbb{Z}_{>0}$.

$$\text{So } \sum_{n=1}^m n(n+1) = \frac{1}{3}m(m+1)(m+2).$$

Then

$$\begin{aligned} \sum_{n=1}^{m+1} n(n+1) &= (m+1)(m+2) + \sum_{n=1}^m n(n+1) \\ &= (m+1)(m+2) + \frac{1}{3}m(m+1)(m+2) \quad (\text{by the induction hypothesis}) \\ &= (m+1)(m+2) \left(1 + \frac{1}{3}m\right) \\ &= \frac{1}{3}(m+1)(m+2)(m+3). \end{aligned}$$

Hence, $P(m+1)$ is true.

Thus, $P(m)$ implies $P(m+1)$ and, since $P(1)$ is true, by induction $P(k)$ is true for all $k \in \mathbb{Z}_{>0}$. ■

5. Let A be the set of all functions from $\mathbb{R}$ to $\mathbb{R}$.

Define a relation on A by

$(f, g) \in R$ iff there exists $k \in \mathbb{R}, k \neq 0$, such that $f(x) = g(kx)$ for all $x \in \mathbb{R}$.

Prove that R is an equivalence relation.

Let $f \in A$.

Then $f(x) = f(1 \cdot x)$ for all $x \in \mathbb{R}$.

Hence, $(f, f) \in R$.

Thus, R is reflexive.

Suppose $(f, g) \in R$.

Then there exists a $k \in \mathbb{R}, k \neq 0$ such that $f(x) = g(kx)$ for all $x \in \mathbb{R}$.

Then $\frac{1}{k} \in \mathbb{R}$. Let $m = \frac{1}{k}$.

Then $f(mx) = g(kmx) = g(x)$ for all $x \in \mathbb{R}$.

Hence, $(g, f) \in R$.

Thus, R is symmetric.

Suppose $(f, g) \in R$ and $(g, h) \in R$.

Then there exist $k_1 \in \mathbb{R}$ and $k_2 \in \mathbb{R}$ with $k_1 \neq 0$ and $k_2 \neq 0$ such that $g(x) = f(k_1x)$ and $h(x) = g(k_2x)$ for all $x \in \mathbb{R}$.

Then $h(x) = f(k_1k_2x)$ for all $x \in \mathbb{R}$.

Let $m = \frac{1}{k_1k_2}$.

Then $m \neq 0$ and $h(mx) = f(x)$ for all $x \in \mathbb{R}$.

Since $k_1k_2 \in \mathbb{R}$ and $k_1k_2 \neq 0$, we conclude that $(f, h) \in R$.

Hence, R is transitive.

Since R is reflexive, symmetric and transitive, R is an equivalence relation. ■

6. Let $A = \mathcal{P}(\mathbb{R})$.

Define $f : \mathbb{R} \rightarrow A$ by

$$f(x) = \{kx : k \in \mathbb{Q}\}.$$

(a) Is f one-to-one? Prove your answer is correct.

If $x = 1$, then $f(x) = \{k : k \in \mathbb{Q}\} = \mathbb{Q}$.

Let $x = -1$. Then $f(x) = \{-k : k \in \mathbb{Q}\}$.

Suppose $a \in f(x)$. Then $a = -k$ for some $k \in \mathbb{Q}$ and so a is rational, i.e. $a \in \mathbb{Q}$.

Hence, $a \in f(x)$ implies $a \in \mathbb{Q}$, so $f(x) \subseteq \mathbb{Q}$.

Now suppose $a \in \mathbb{Q}$. Then $a = -(-a)$ and $-a \in \mathbb{Q}$, so $a \in f(x)$.

Hence, $a \in \mathbb{Q}$ implies $a \in f(x)$, so $\mathbb{Q} \subseteq f(x)$.

Thus, $\mathbb{Q} = f(x)$.

Hence, $f(1) = f(-1)$ and so f is not one-to-one. ■

(b) Is f onto? Prove your answer is correct.

Let $x \in \mathbb{R}$.

Then $0 \in f(x)$ since $0 = (0)x$ and $0 \in \mathbb{Q}$.

Hence, for all x , $0 \in f(x)$.

Consider $\{1\} \in A$.

Since 0 is not in this set, there is no $x \in \mathbb{R}$ such that $f(x) = \{1\}$.

Thus, f is not onto. ■

(There are quite a number of possible proofs here. One alternative is to show that $f(x)$ is never empty by showing it always contains x and hence there is no x such that $f(x) = \emptyset$, while $\emptyset \in A$.)