

Math 300 C - Spring 2022
Final Exam
June 9, 2022
Solutions

1. Let A , B , and C be sets.

Prove that $A \cup C \subseteq B \cup C$ iff $A \setminus C \subseteq B \setminus C$.

Suppose $A \cup C \subseteq B \cup C$.

Suppose $x \in A \setminus C$.

Then $x \in A$ and $x \notin C$.

So $x \in A \cup C$ so $x \in B \cup C$ since $A \cup C \subseteq B \cup C$.

Since $x \notin C$, $x \in B$.

Hence, $x \in B \setminus C$.

Thus, $x \in A \setminus C$ implies $x \in B \setminus C$, so $A \setminus C \subseteq B \setminus C$.

Suppose $A \setminus C \subseteq B \setminus C$.

Suppose $x \in A \cup C$.

Suppose $x \in C$.

Then $x \in B \cup C$.

Suppose $x \notin C$.

Then $x \in A$ and $x \notin C$, so $x \in A \setminus C$.

Hence, $x \in B \setminus C$, since $A \setminus C \subseteq B \setminus C$.

So, $x \in B$, and so $x \in B \cup C$.

Hence, $x \in A \cup C$ implies $x \in B \cup C$, so $A \cup C \subseteq B \cup C$.

Therefore, $A \cup C \subseteq B \cup C$ iff $A \setminus C \subseteq B \setminus C$. ■

2. Let A be the set of all functions $f : \mathbb{R} \rightarrow \mathbb{R}$. Define a relation R on A by

$(f, g) \in R$ iff there exists a $c \in \mathbb{R}$ such that $f(x) = g(x) + c$ for all $x \in \mathbb{R}$.

Prove that R is an equivalence relation.

Let $f \in A$.

Then $f(x) = f(x) + 0$ for all $x \in \mathbb{R}$.

Hence, $(f, f) \in R$.

Thus, R is reflexive.

Suppose $(f, g) \in R$.

Then there exists $c \in \mathbb{R}$ such that $f(x) = g(x) + c$ for all $x \in \mathbb{R}$.

Then $g(x) = f(x) + (-c)$ for all $x \in \mathbb{R}$.

Hence, $(g, f) \in R$ since $-c \in \mathbb{R}$.

Thus, R is symmetric.

Suppose $(f, g) \in R$ and $(g, h) \in R$.

Then there exists $c \in \mathbb{R}$ such that $f(x) = g(x) + c$ for all $x \in \mathbb{R}$.

Also, there exists $d \in \mathbb{R}$ such that $g(x) = h(x) + d$ for all $x \in \mathbb{R}$.

Then, $f(x) = h(x) + (c + d)$ for all $x \in \mathbb{R}$.

Hence, $(f, h) \in R$ since $c + d \in \mathbb{R}$.

Thus, R is transitive.

Since R is reflexive, symmetric and transitive, R is an equivalence relation. ■

3. (a) Let $\mathcal{F}$ and $\mathcal{G}$ be families. Prove that $\bigcup \mathcal{F} \setminus \bigcup \mathcal{G} \subseteq \bigcup (\mathcal{F} \setminus \mathcal{G})$.

Suppose $x \in \bigcup \mathcal{F} \setminus \bigcup \mathcal{G}$.

Then $x \in \bigcup \mathcal{F}$ and $x \notin \bigcup \mathcal{G}$.

Hence, there exists $S \in \mathcal{F}$ such that $x \in S$.

Also, $S \notin \mathcal{G}$ since $x \notin \bigcup \mathcal{G}$.

Hence, $S \in \mathcal{F} \setminus \mathcal{G}$, and thus $x \in \bigcup (\mathcal{F} \setminus \mathcal{G})$.

Thus, $x \in \bigcup \mathcal{F} \setminus \bigcup \mathcal{G}$ implies $x \in \bigcup (\mathcal{F} \setminus \mathcal{G})$.

Therefore, $\bigcup \mathcal{F} \setminus \bigcup \mathcal{G} \subseteq \bigcup (\mathcal{F} \setminus \mathcal{G})$. ■

- (b) Prove that there exist non-empty families $\mathcal{F}$ and $\mathcal{G}$ such that $\bigcup \mathcal{F} \setminus \bigcup \mathcal{G} \neq \bigcup (\mathcal{F} \setminus \mathcal{G})$.

Let $\mathcal{F} = \{\{1, 2\}\}$ and $\mathcal{G} = \{\{1\}\}$.

Then $\bigcup \mathcal{F} = \{1, 2\}$ and $\bigcup \mathcal{G} = \{1\}$.

So, $\bigcup \mathcal{F} \setminus \bigcup \mathcal{G} = \{2\}$.

Also, $\mathcal{F} \setminus \mathcal{G} = \mathcal{F}$ so

$$\bigcup (\mathcal{F} \setminus \mathcal{G}) = \{1, 2\} \neq \{2\} = \bigcup \mathcal{F} \setminus \bigcup \mathcal{G}.$$

Hence, there exist non-empty families $\mathcal{F}$ and $\mathcal{G}$ such that $\bigcup \mathcal{F} \setminus \bigcup \mathcal{G} \neq \bigcup (\mathcal{F} \setminus \mathcal{G})$. ■

4. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 - 2x + 3 & \text{if } x \leq 1; \\ -3x + 5 & \text{if } x > 1. \end{cases}$$

Prove that f is a bijection.

We first note that $x^2 - 2x + 3 = (x - 1)^2 + 2$.

Since $(x - 1)^2 \geq 0$ for all $x \in \mathbb{R}$, we conclude that $x^2 - 2x + 3 \geq 2$ for all $x \in \mathbb{R}$.

Hence $f(x) \geq 2$ for all $x \leq 1$.

Also, if $x > 1$, then $-3x < -3$ so $-3x + 5 < 2$, i.e., $f(x) < 2$.

Let $y \in \mathbb{R}$.

Suppose $y \geq 2$.

Let $a = 1 - \sqrt{y - 2}$.

Then $a \leq 1$, so $f(a) = (a - 1)^2 + 2 = y - 2 + 2 = y$.

Suppose $y < 2$.

Let $a = -\frac{1}{3}(y - 5)$.

Since $y < 2$, $y - 5 < -3$ and $a = -\frac{1}{3}(y - 5) > 1$ and so $f(a) = -3a + 5 = y - 5 + 5 = y$.

Thus, for all $y \in \mathbb{R}$, there exists $a \in \mathbb{R}$ such that $f(a) = y$, i.e., f is surjective.

Suppose $f(x_1) = f(x_2)$ for some $x_1, x_2 \in \mathbb{R}$.

Suppose $f(x_1) < 2$.

Then $f(x_2) < 2$ and so $f(x_1) = -3x_1 + 5 = -3x_2 + 5$.

Solving $-3x_1 + 5 = -3x_2 + 5$ we find $x_1 = x_2$.

Suppose $f(x_1) \geq 2$.

Then $f(x_2) \geq 2$ and $f(x_1) = (x_1 - 1)^2 + 2 = (x_2 - 1)^2 + 2$ and $x_1 \leq 1$ and $x_2 \leq 1$.

Hence, $(x_1 - 1)^2 = (x_2 - 1)^2$, so $|x_1 - 1| = |x_2 - 1|$.

Since $x_1 \leq 1$ and $x_2 \leq 1$, $|x_1 - 1| = 1 - x_1$ and $|x_2 - 1| = 1 - x_2$.

So we conclude $1 - x_1 = 1 - x_2$ and thus $x_1 = x_2$.

Thus $f(x_1) = f(x_2)$ implies $x_1 = x_2$: f is injective.

Since f is injective and surjective, f is a bijection. ■

5. Prove that $20! + 13$ cannot be written as $i^2 + j^4$ for any integers i and j . (Hint: consider the situation modulo 5).

By Euclid's Division theorem, every integer is congruent to 0, 1, 2, 3 or 4 modulo 5.

Considering squares and fourth powers modulo 5, we have these possibilities (note we can use the fact that $n^4 = (n^2)^2$ here.)

n	$n^2(\text{mod } 5)$	$n^4(\text{mod } 5)$
0	0	0
1	1	1
2	4	$16 \equiv 1$
3	$9 \equiv 4$	1
4	1	1

Thus, for any integer n , $n^2 \equiv 0, 1$ or $4 \pmod{5}$ and $n^4 \equiv 0$ or $1 \pmod{5}$.

We can calculate all possible sums modulo 5 of a square and a fourth power in this modulo 5 addition table:

$+$	0	1	4
0	0	1	4
1	1	2	0

Thus, we see that, for any integers i and j , $i^2 + j^4 \equiv 0, 1, 2$ or $4 \pmod{5}$.

So $i^2 + j^4$ cannot be congruent to 3 (mod 5) for any integers i and j .

Now, $20! + 13 = 20(19!) + 10 + 3 = 5(4(19!) + 2) + 3 \equiv 3 \pmod{5}$.

Suppose $20! + 13 = i^2 + j^4$ for some integers i and j .

Then $i^2 + j^4 \equiv 3 \pmod{5}$.

This is a contradiction to our earlier statement that such a sum cannot be congruent to 3 modulo 5.

Hence, there do not exist integers i and j such that $i^2 + j^4 = 20! + 13$. ■

6. Suppose A , B and C are sets. Suppose $f : A \rightarrow B$ and $g : B \rightarrow C$.

(a) Prove that if f is onto and g is not one-to-one, then $g \circ f$ is not one-to-one.

Suppose f is onto and g is not one-to-one.

Then there exist $x_1, x_2 \in B$ such that $g(x_1) = g(x_2)$ and $x_1 \neq x_2$.

Since f is onto, there exist $a_1, a_2 \in A$ such that $f(a_1) = x_1$ and $f(a_2) = x_2$.

Since $f(a_1) \neq f(a_2)$, we conclude $a_1 \neq a_2$.

Then $g(f(a_1)) = g(f(a_2))$ with $a_1 \neq a_2$, so g is not one-to-one. ■

(b) Give an example to show that $g \circ f$ may be one-to-one when g is not one-to-one.

Let $A = \{a\}$, $B = \{b_1, b_2\}$, $C = \{y\}$.

Let $f = \{(a, b_1)\}$ and $g = \{(b_1, y), (b_2, y)\}$.

Then g is not one-to-one: $g(b_1) = g(b_2)$ but $b_1 \neq b_2$.

Also, $g \circ f = \{(a, c)\}$, so $g \circ f$ is one-to-one. ■

Elementary Properties of the Integers (EPIs)

Suppose a, b, c , and d are integers.

1. **Closure:** $a + b$ and ab are integers.
2. **Substitution of Equals:** If $a = b$, then $a + c = b + c$ and $ac = bc$.
3. **Commutativity:** $a + b = b + a$ and $ab = ba$.
4. **Associativity:** $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$.
5. **The Distributive Law:** $a(b + c) = ab + ac$
6. **Identities:** $a + 0 = 0 + a = a$ and $a \cdot 1 = 1 \cdot a = a$.
0 is called the additive identity.
1 is called the multiplicative identity.
7. **Additive Inverses:** There exists an integer $-a$ such that $a + (-a) = (-a) + a = 0$.
8. **Trichotomy:** Exactly one of the following is true:
 $a > 0$, $-a > 0$, or $a = 0$.
9. **The Well-Ordering Principle:** Every non-empty set of positive integers contains a smallest element.
10. $a \cdot 0 = 0$
11. If $a + c = b + c$, then $a = b$.
12. $-a = (-1) \cdot a$
13. $(-a) \cdot b = -(ab)$
14. $(-a) \cdot (-b) = ab$
15. If $ab = 0$, then $a = 0$ or $b = 0$.
16. If $a \leq b$ and $b \leq a$, then $a = b$.
17. If $a < b$ and $b < c$, then $a < c$.
18. If $a < b$, then $a + c < b + c$.
19. If $a < b$ and $0 < c$, then $ac < bc$.
20. If $a < b$ and $c < 0$, then $bc < ac$.
21. If $a < b$ and $c < d$, then $a + c < b + d$.
22. If $0 \leq a < b$ and $0 \leq c < d$, then $ac < bd$.
23. If $a < b$, then $-b < -a$.
24. $0 \leq a^2$, where $a^2 = a \cdot a$.
25. If $ab = 1$, then either $a = b = 1$ or $a = b = -1$.

Sets

- $A \subseteq B$ iff $x \in A$ implies $x \in B$
 $A = B$ iff $A \subseteq B$ and $B \subseteq A$
 $x \in A \cup B$ iff $x \in A$ or $x \in B$
 $x \in A \cap B$ iff $x \in A$ and $x \in B$
 $x \in A \setminus B$ iff $x \in A$ and $x \notin B$

NOTE: Properties 17-23 hold if each $<$ is replaced with $\leq$.

One theorem for reference:

Theorem DAS (Divisors are Smaller):

Let a and b be positive integers. Then $a|b$ implies $a \leq b$.