1. Let $S$ be a set.

Define a function $f : \mathcal{P}(S) \to \mathcal{P}(S)$ by $f(A) = S \setminus A$ for all $A \in \mathcal{P}(S)$.

Prove that $f$ is a bijection.

**Proof:** Suppose $A_1$ and $A_2 \in \mathcal{P}(S)$ with $f(A_1) = f(A_2)$. Then

$$S \setminus A_1 = S \setminus A_2.$$ 

Suppose $x \in A_1$ but $x \not\in A_2$. Then $x \in S$ (since $A_1 \subset S$), and $x \in S \setminus A_2$, but $x \not\in S \setminus A_1$. This is a contradiction to the fact that $S \setminus A_1 = S \setminus A_2$. Hence if $x \in A_1$ then $x \in A_2$; i.e., $A_1 \subset A_2$.

Suppose $x \in A_2$ but $x \not\in A_1$. Then $x \in S$ (since $A_2 \subset S$), and $x \in S \setminus A_1$, but $x \not\in S \setminus A_2$. This is a contradiction to the fact that $S \setminus A_2 = S \setminus A_1$. Hence if $x \in A_2$ then $x \in A_1$; i.e., $A_2 \subset A_1$.

Thus $A_1 = A_2$, so $f$ is one-to-one.

Suppose $B \in \mathcal{P}(S)$.

Let $Z = S \setminus B$.

Then

$$f(Z) = S \setminus (S \setminus B)$$

$$= \{x \in S : x \not\in (S \setminus B)\}$$

$$= \{x \in S : \neg(x \in S \setminus B)\}$$

$$= \{x \in S : \neg(x \in S \land x \not\in B)\}$$

$$= \{x \in S : x \not\in S \lor x \in B\}$$

$$= \{x \in B\}$$

$$= B.$$ 

Thus $f$ is onto.

Hence $f$ is one-to-one and onto, i.e., it is a bijection.
2. Let \( S \) be the set of all functions \( f : \mathbb{R} \to \mathbb{R} \). Define a relation \( R \) on \( S \) by

\[
(f, g) \in R \iff \exists c \in \mathbb{R}, c \neq 0, \text{ such that } f(x) = cg(x) \text{ for all } x \in \mathbb{R}.
\]

(a) Prove that \( R \) is an equivalence relation.

(b) Let \( f : \mathbb{R} \to \mathbb{R} \) be a non-constant, linear function.

Consider \([f]\), the equivalence class of \( f \) under the relation \( R \).

Let \( g : \mathbb{R} \to \mathbb{R} \) be a linear function.

Show that, \( g \in [f] \iff f \) and \( g \) have the same \( x \)-intercept.

(a) Proof: Let \( f \in S \).

Since \( f(x) = (1)f(x) \) for all \( x \in \mathbb{R} \), \( (f, f) \in R \).

Hence, \( R \) is reflexive.

Suppose \((f, g) \in R \).

Then there exists a \( c \in \mathbb{R}, c \neq 0 \) such that \( f(x) = cg(x) \) for all \( x \in \mathbb{R} \). Since \( c \neq 0 \),

\[
g(x) = \frac{1}{c} f(x)
\]

for all \( x \in \mathbb{R} \) and \( \frac{1}{c} \neq 0, \frac{1}{c} \in \mathbb{R} \).

Thus \((g, f) \in R \).

Hence, \( R \) is symmetric.

Suppose \((f, g) \in R \) and \((g, h) \in R \).

Then there exist non-zero \( c, d \in \mathbb{R} \) such that \( f(x) = cg(x) \) and \( g(x) = dh(x) \) for all \( x \in \mathbb{R} \).

Hence, \( f(x) = cdh(x) \) for all \( x \in \mathbb{R} \).

Since \( c \) and \( d \) are non-zero, \( cd \) is non-zero, and \( cd \in \mathbb{R} \), so \((f, h) \in R \).

Thus, \( R \) is transitive, and so \( R \) is an equivalence relation. ■

(b) Proof: Suppose \( g \in [f] \).

Then \((g, f) \in R \) so there exists a non-zero \( c \in \mathbb{R} \) such that \( g(x) = cf(x) \) for all \( x \in \mathbb{R} \).

Since \( g \) is non-constant, it has an \( x \)-intercepts; suppose \( g(z) = 0 \).

Then \( 0 = g(z) = cf(z) \), and since \( c \neq 0 \), \( f(z) = 0 \).

Hence, \( g \) and \( f \) have the same \( x \)-intercept.

Now, suppose \( g \) and \( f \) have the same \( x \)-intercept; suppose \( g(a) = f(a) = 0 \).

Suppose \( g(x) = m_1 x + b_1 \) and \( f(x) = m_2 x + b_2 \).

Then \( m_1 a + b_1 = 0 \) and \( m_2 a + b_2 = 0 \), so

\[
a = -\frac{b_1}{m_1} = -\frac{b_2}{m_2},
\]

so \( b_1 = \frac{m_1 b_2}{m_2} \).

Let \( c = \frac{m_1}{m_2} \).

Then \( cf(x) = \frac{m_1}{m_2} (m_2 x + b_2) = m_1 x + b_2 \frac{m_1}{m_2} = m_1 x + b_1 = g(x) \) for all \( x \in \mathbb{R} \).

Hence \( g \in [f] \). ■
3. Let $A$, $B$, and $C$ be sets.

Prove that $A \cup C \subseteq B \cup C$ iff $A \setminus C \subseteq B \setminus C$.

**Proof:** Suppose $A \cup C \subseteq B \cup C$.

Suppose $x \in A \setminus C$.

Then $x \in A$, so $x \in A \cup C$, and since $A \cup C \subseteq B \cup C$, $x \in B \cup C$.

Since $x \in A \setminus C$, $x \notin C$.

Since $x \in B \cup C$, but $x \notin C$, $x \in B$ and so $x \in B \setminus C$.

Hence $A \setminus C \subseteq B \setminus C$.

Now, suppose $A \setminus C \subseteq B \setminus C$.

Suppose $x \in A \cup C$.

If $x \in C$ then $x \in B \cup C$.

Suppose $x \notin C$.

Then $x \in A$ and so $x \in A \setminus C$.

Since $A \setminus C \subseteq B \setminus C$, $x \in B \setminus C$, and so $x \in B$, and hence $x \in B \cup C$.

Thus, $A \cup C \subseteq B \cup C$. ■

4. Let $A$ and $B$ be sets.

Let $f$ and $g$ be functions from $A$ to $B$.

Prove that if $f \cap g \neq \emptyset$, then $f \setminus g$ is not a function from $A$ to $B$.

**Proof:** Suppose $f \cap g \neq \emptyset$.

Suppose $(a, b) \in f \cap g$ (note that this is the unique pair in $f$ with first element $a$).

Then $(a, b) \notin f \setminus g$.

Since $f$ is a function, there is no element $(x, y) \in f \setminus g$ such that $x = a$.

Hence $f \setminus g$ is not a function. ■

5. Let $n \in \mathbb{Z}_{>0}$.

Use induction to prove $\sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}$.

**Proof:** Let $P(n)$ be the statement $\sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}$.

Since $\sum_{i=1}^{1} \frac{1}{(2i-1)(2i+1)} = \frac{1}{3} = \frac{1}{2(1)+1}$.

P(1) is true.

Suppose there exists a $k > 0$ such that $P(k)$ is true.
Then

\[
\sum_{i=1}^{k+1} \frac{1}{(2i-1)(2i+1)} = \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} \quad \text{(by the induction hypothesis)}
\]

\[
= \frac{k(2k+3) + 1}{(2k+1)(2k+3)}
\]

\[
= \frac{k+1}{2k+3}
\]

Hence \(P(k+1)\) is true, so \(P(k)\) implies \(P(k+1)\).
Hence, by induction, \(P(k)\) is true for all \(k > 0\). □

6. Let \(n\) be an integer. Prove that \(4|n^4 + 2n\) iff \(n\) is even.

**Proof:**

Suppose \(n\) is even.

Then \(n = 2m\) for some integer \(m\), and

\[n^4 + 2n = (2m)^4 + 2(2m) = 16m^4 + 4m = 4(4m^4 + m)\]

and so \(4|n^4 + 2n\).

Now suppose \(n\) is odd.

Then \(n = a + 1\) for some even integer \(a\), and

\[n^4 + 2n = (a + 1)^4 + 2(a + 1) = a^4 + 4a^3 + 6a^2 + 4a + 1 + 2a + 1\]

\[\equiv a^4 + 6a^2 + 2a + 2 \pmod{4}\]

\[\equiv 6a^2 + 3 \pmod{4} \quad \text{(since \(a\) is even, so \(4|a^4 + 2a\))}\]

\[\equiv 2a^2 + 3 \pmod{4}.\]

As was shown in a homework assignment, \(a^2\) is congruent to 0 or 1 mod 4.
Thus, \(2a^2\) is congruent to 0 or 2 mod 4.
Hence, \(2a^2 + 3\) is congruent to 3 or 1 mod 4; hence 4 does not divide \(n^4 + 2n\) when \(n\) is odd.

□