1. Assign “true” or “false” to each of the following statements. No justification need be given.

   (a) T
   (b) F
   (c) F
   (d) T
   (e) T
   (f) F
   (g) T
   (h) F

2. Suppose \( f : A \to B \) and \( f \) is one-to-one. Prove that there is some set \( B' \subseteq B \) such that \( f^{-1} : B' \to A \).

   Let \( B' = \{ b \in B : \exists a \in A \text{ such that } f(a) = b \} \). Consider \( g : A \to B' \) defined by \( g(a) = f(a) \) for \( a \in A \).

   Let \( b \in B' \). Then \( \exists a \in A \) with \( g(a) = b \). So \( g \) is onto.

   Suppose \( a_1, a_2 \in A \) and \( g(a_1) = g(a_2) \). Then \( f(a_1) = f(a_2) \), so \( a_1 = a_2 \) since \( f \) is one-to-one. Hence \( g \) is one-to-one.

   Thus \( g \) is a bijection, so \( g^{-1} : B' \to A \).

   But, \( g^{-1} = f^{-1} \), so \( f^{-1} : B' \to A \).

3. Let \( A = \mathcal{P}(\mathbb{R}) \). Define \( f : \mathbb{R} \to A \) by the formula

   \[ f(x) = \{ y \in \mathbb{R} : y^2 < x \} \]

   (a) Is \( f \) one-to-one? Prove your answer.

   Since \( f(0) = \emptyset \), and \( f(-1) = \emptyset \), and \( 0 \neq -1 \), \( f \) is not one-to-one.

   (b) Is \( f \) onto? Prove your answer.

   No.

   Suppose \( a > 0 \). Then \( f(a) = (-\sqrt{a}, \sqrt{a}) \). So, \( 0 \in f(a) \).

   Now, suppose \( a < 0 \). Then \( f(a) = \emptyset \).

   Hence, \( f(a) \) is either the empty set, or \( f(a) \) contains zero.

   Consider \( T = (1, 2) \in A \). We know that \( T \not\in \emptyset \) and \( 0 \not\in T \). Hence, \( T \neq f(a) \) for any \( a \in \mathbb{R} \).

   Thus, \( f \) is not onto.

4. Let \( S = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x - y \in \mathbb{Z} \} \).

   Is \( S \) an equivalence relation? Prove your answer.

   Yes.
Reflexive:
For all $x \in \mathbb{R}$, $x - x = 0 \in \mathbb{Z}$. So, $(x, x) \in R$ for all $x \in \mathbb{R}$. Hence, $R$ is reflexive.

Symmetric:
Suppose $x, y \in \mathbb{R}$ and $x - y = m \in \mathbb{Z}$.
Then $y - x = -m \in \mathbb{Z}$.
So, $(x, y) \in R$ implies $(y, x) \in R$. Thus, $R$ is symmetric.

Transitive:
Suppose $(x, y) \in R$ and $(y, z) \in R$.
Let $x - y = a$ and $y - z = b$. Then $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$.
Then $x - z = a + b \in \mathbb{Z}$ since $\mathbb{Z}$ is closed under addition.
Hence $(x, z) \in R$, and so $R$ is transitive.
Thus $R$ is an equivalence relation.

5. Use induction to prove that $49$ divides $36^n + 14n - 1$ for all $n \in \mathbb{Z}_{\geq 0}$.

If $n = 0$, then $36^n + 14n - 1 = 1 + 0 - 1 = 0$, so $49|36^n + 14n - 1$.

Suppose $49|36^k + 14k - 1$ for some $k \in \mathbb{Z}_{\geq 0}$. Say $36^k + 14k - 1 = 49m$ for some $m \in \mathbb{Z}$.
Then

$$36^{k+1} + 14(k + 1) - 1 =$$
$$36(36^k + 14k - 1) - 36(14k) + 36 + 14(k + 1) - 1 =$$
$$36(49m) - 36(14k) + 14k + 36 + 14 - 1 =$$
$$36(49m) - 35(14k) + 49 =$$
$$36(49m) - 49(10k) + 49 =$$
$$49(36m - 10k + 1).$$

Hence, $49|36^{k+1} + 14(k + 1) - 1$.
Thus, $49|36^n + 14(n) - 1$ for all $n \in \mathbb{Z}_{\geq 0}$.

6. Suppose $R$ is an equivalence relation on a set $A$.

Prove that for every $x \in A$ and $y \in A$, $y \in [x]_R$ iff $[y]_R = [x]_R$.

($\leftarrow$)
Suppose $[y]_R = [x]_R$. Since $y \in [y]_R$, $y \in [x]_R$.

($\rightarrow$)
Suppose $y \in [x]_R$. Then $(x, y) \in R$. Let $a \in [y]_R$. Then $(a, y) \in R$.
But $(x, y) \in R$, so $(y, x) \in R$, so by transitivity of $R$, $(a, x) \in R$. Hence, $a \in [x]$. Thus, $[y]_R \subseteq [x]_R$.
Let $b \in [x]_R$. Then $(x, b) \in R$. But, $(y, x) \in R$. By transitivity, $(y, b) \in R$, so $b \in [y]_R$.
So, $[x]_R \subseteq [y]_R$.
Thus, $[x]_R = [y]_R$. 