1. Assign “true” or “false” to each of the following statements. No justification need be given.

(a) F
(b) T
(c) T
(d) T
(e) F
(f) F
(g) T
(h) T

2. Let $A = \mathcal{P}(\mathbb{R})$. Define $f : \mathbb{R} \rightarrow A$ by the formula

$$f(x) = \{y \in \mathbb{R} : y^2 < x\}.$$ 

(a) Is $f$ one-to-one? Prove your answer.
Since $f(0) = \emptyset$, and $f(-1) = \emptyset$, and $0 \neq -1$, $f$ is not one-to-one.

(b) Is $f$ onto? Prove your answer.
No.
Suppose $a > 0$. Then $f(a) = (-\sqrt{a}, \sqrt{a})$. So, $0 \in f(a)$.
Now, suppose $a < 0$. Then $f(a) = \emptyset$.
Hence, $f(a)$ is either the empty set, or $f(a)$ contains zero.
Consider $T = (1, 2) \in A$. We know that $T \not\in \emptyset$ and $0 \not\in T$. Hence, $T \neq f(a)$ for any $a \in \mathbb{R}$.
Thus, $f$ is not onto.

3. Let $R$ be a relation on $\mathbb{Q}$ defined by $(p/q, r/s) \in R \iff ps = qr$. Show that $R$ is an equivalence relation.
Reflexive:
For any $p/q \in \mathbb{Q}$, $pq = qp$, $(p/q, p/q) \in R$. Hence, $R$ is reflexive.
Symmetric:
Suppose $(a/b, c/d) \in R$. Then $ad = bc$. Hence, $bc = ad$ and $cb = da$, so $(c/d, a/b) \in R$. Thus, $R$ is symmetric.
Transitive:
Suppose $(a_1/b_1, a_2/b_2) \in R$ and $(a_2/b_2, a_3/b_3) \in R$.
Then $a_1b_2 = a_2b_1$ and $a_2b_3 = a_3b_2$.
Then $a_1b_2b_3 = a_2b_1b_3$ so
\[a_1b_3b_2 = b_1a_2b_3\]
so
\[a_1b_3b_2 = b_1a_3b_2\]
and hence
\[a_1b_3 = b_1a_3.\] Hence, \((a_1/b_1, a_3/b_3) \in R.\) Thus, \(R\) is transitive.
Since \(R\) is reflexive, symmetric and transitive, \(R\) is an equivalence relation.

4. Give a proof by induction that \(6\) divides \(n^3 - n\) for all \(n \in \mathbb{Z}_{\geq 0}\).
Let \(n = 0\). Then \(n^3 - n = 0\) and \(6|0\), so \(6|n^3 - n\) when \(n = 0\).
Suppose \(6|k^3 - k\) for some \(k \geq 0\).
Then
\[
(k + 1)^3 - (k + 1) = \\
k^3 + 3k^2 + 3k + 1 - k - 1 = \\
k^3 + 3k^2 + 2k = \\
k^3 - k + 3k^2 + 3k = \\
k^3 - k + 3(k^2 + k) = \\
k^3 - k + 3k(k + 1).
\]
For all \(k \in \mathbb{Z}\), \(k\) is even or \(k + 1\) is even. Hence, \(k(k + 1)\) is even, and so \(6|3k(k + 1)\).
Since \(6|k^3 - k, 6|(k + 1)^3 - (k + 1)\).
Hence, \(6|n^3 - n\) for all \(n \geq 0\).

5. Suppose \(f : A \to C\) and \(g : B \to C\). Prove that if \(A\) and \(B\) are disjoint, then
\[f \cup g : A \cup B \to C.\]
Let \(x \in A \cup B\). Since \(A \cap B = \emptyset\), \(x \in A\) or \(x \in B\) and not both.
If \(x \in A\), then \(\exists c \in C\) such that \((x, c) \in f\).
If \(x \in B\), then \(\exists c \in C\) such that \((x, c) \in g\).
Thus, \(\exists c \in C\) such that \((x, c) \in f \cup g\).
Suppose \(\exists c_1, c_2 \in C\) and \(x \in A \cup B\) such that
\[(x, c_1) \in f \cup g \text{ and } (x, c_2) \in f \cup g.\]
Then three cases are possible:

1. \((x, c_1) \in f \text{ and } (x, c_2) \in f\)
2. \((x, c_1) \in f \text{ and } (x, c_2) \in g\)
3. \((x, c_1) \in g \text{ and } (x, c_2) \in g\)

Case (1) would imply that \(f\) is not a function.
Case (3) would imply that $g$ is not a function.
Case (2) would imply that $x \in A$ and $x \in B$, which is not possible since $A \cap B = \emptyset$.
Thus, for all $x \in A \cup B$, there exists a unique $c \in C$ such that $(x, c) \in f \cup g$.
That is, $f \cup g : A \cup B \to C$.


Let’s show $R \subseteq S$.
Suppose $(a, b) \in R$.
Then $\exists x \in A/R$ such that $a \in x$ and $b \in x$.
Hence, $x \in A/S$ and $a \in x$ and $b \in x$.
Thus, $(a, b) \in S$ and so $R \subseteq S$.
A symmetric argument shows that $S \subseteq R$, and so $R = S$. 