

Math 300 C - Autumn 2025  
Final Exam  
December 9, 2025

Name: \_\_\_\_\_

Student ID no. : \_\_\_\_\_

Signature: \_\_\_\_\_

|       |    |
|-------|----|
| 1     | 10 |
| 2     | 10 |
| 3     | 10 |
| 4     | 10 |
| 5     | 10 |
| 6     | 10 |
| Total | 60 |

- You have 110 minutes to complete the exam.
- When time is called, you must immediately stop writing.
- No calculators or other devices are allowed during the exam.
- No notesheets or other information sources are allowed during the exam.
- The last page of the exam lists the EPIs and other facts for your use.
- You should not use any result from homework, etc., without first proving it on this exam, unless otherwise indicated.

1. Prove the following theorem.

**Theorem:** Let  $A$  and  $B$  be non-empty sets.

Let  $f : A \rightarrow B$ .

Suppose  $g \subseteq f$  and  $g \neq \emptyset$ .

Then there exists a non-empty set  $A'$  such that  $g : A' \rightarrow B$ .

2. (a) Suppose  $A$  is a countably infinite set and  $b \notin A$ .  
Prove that  $A \cup \{b\}$  is countably infinite by demonstrating the necessary bijection.  
You need not prove it is a bijection.

(b) Prove that the real intervals  $(-2, 3)$  and  $(5, 10)$  are equinumerous.

3. Let  $A$  and  $B$  be sets.

(a) Prove that  $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$ .

(b) Prove that there exist sets  $A$  and  $B$  such that  $\mathcal{P}(A \cup B) \not\subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$ .

4. Prove that, if  $n$  is an integer and  $3|n$  and  $5|n$ , then  $15|n$ .

5. Let  $A$  be the set of all real functions  $f : \mathbb{R} \rightarrow \mathbb{R}$ . Define a relation  $R$  on  $A$  by

$$(f, g) \in R \text{ iff } \exists k \in \mathbb{R} \text{ such that } f(x) = g(x) + k \text{ for all } x \in \mathbb{R}.$$

Prove that  $R$  is an equivalence relation.

6. Use induction to prove that, for all positive integers  $n$ ,

$$\sum_{j=1}^n \frac{1}{j(j+1)} = \frac{n}{n+1}.$$

## Elementary Properties of the Integers (EPIs)

Suppose  $a, b, c$ , and  $d$  are integers.

- **Closure:**  $a + b$  and  $ab$  are integers.
- **Substitution of Equals:** If  $a = b$ , then  $a + c = b + c$  and  $ac = bc$ .
- **Commutativity:**  $a + b = b + a$  and  $ab = ba$ .
- **Associativity:**  $(a + b) + c = a + (b + c)$  and  $(ab)c = a(bc)$ .
- **The Distributive Law:**  $a(b + c) = ab + ac$
- **Identities:**  $a + 0 = 0 + a = a$  and  $a \cdot 1 = 1 \cdot a = a$ .  
 $0$  is called the additive identity.  
 $1$  is called the multiplicative identity.
- **Additive Inverses:** There exists an integer  $-a$  such that  $a + (-a) = (-a) + a = 0$ .
- **Trichotomy:** Exactly one of the following is true:  
 $a > 0$ ,  $-a > 0$ , or  $a = 0$ .
- **The Well-Ordering Principle:** Every non-empty set of positive integers contains a smallest element.

1.  $a \cdot 0 = 0$
2. If  $a + c = b + c$ , then  $a = b$ .
3.  $-a = (-1) \cdot a$
4.  $(-a) \cdot b = -(ab)$
5.  $(-a) \cdot (-b) = ab$
6. If  $ab = 0$ , then  $a = 0$  or  $b = 0$ .
7. If  $a \leq b$  and  $b \leq a$ , then  $a = b$ .
8. If  $a < b$  and  $b < c$ , then  $a < c$ .
9. If  $a < b$ , then  $a + c < b + c$ .
10. If  $a < b$  and  $0 < c$ , then  $ac < bc$ .
11. If  $a < b$  and  $c < 0$ , then  $bc < ac$ .
12. If  $a < b$  and  $c < d$ , then  $a + c < b + d$ .
13. If  $0 \leq a < b$  and  $0 \leq c < d$ , then  $ac < bd$ .
14. If  $a < b$ , then  $-b < -a$ .
15.  $0 \leq a^2$ , where  $a^2 = a \cdot a$ .
16. If  $ab = 1$ , then either  $a = b = 1$  or  $a = b = -1$ .

### Sets

$A \subseteq B$  iff  $x \in A$  implies  $x \in B$

$A = B$  iff  $A \subseteq B$  and  $B \subseteq A$

$x \in A \cup B$  iff  $x \in A$  or  $x \in B$

$x \in A \cap B$  iff  $x \in A$  and  $x \in B$

$x \in A \setminus B$  iff  $x \in A$  and  $x \notin B$

$\mathcal{P}(A)$  is the set of all subsets of a set  $A$

For a family  $\mathcal{F}$ :

$$\bigcup \mathcal{F} = \{x : \exists S \in \mathcal{F} \text{ such that } x \in S\}$$

$$\bigcap \mathcal{F} = \{x : \forall S \in \mathcal{F}, x \in S\}$$

NOTE: Properties 8-14 hold if each  $<$  is replaced with  $\leq$ .

**Euclid's Division Theorem:** For all positive integers  $m$  and all integers  $n$ , there exist unique integers  $q$  and  $r$ , with  $0 \leq r < m$  such that  $n = qm + r$ .

**Bijections have inverse functions Theorem:** Let  $f : A \rightarrow B$  be a bijection, where  $A$  and  $B$  are sets. Then  $f^{-1} : B \rightarrow A$  and is a bijection.

**Inverse composition Theorem:** A function  $f : A \rightarrow B$  is a bijection iff there is a function  $g : B \rightarrow A$  such that  $f \circ g = i_B$  and  $g \circ f = i_A$ .