

Math 300 C - Autumn 2025  
Final Exam  
December 9, 2025  
Solutions

1. *Prove the following theorem.*

**Theorem:** *Let  $A$  and  $B$  be non-empty sets.*

*Let  $f : A \rightarrow B$ .*

*Suppose  $g \subseteq f$  and  $g \neq \emptyset$ .*

*Then there exists a non-empty set  $A'$  such that  $g : A' \rightarrow B$ .*

**Proof:** Let  $A, B, f$  and  $g$  be as above.

Since  $f \subseteq A \times B$ , and  $g \subseteq f$ , we conclude that  $g \subseteq A \times B$ .

Let  $A' = \{a : (a, b) \in g\}$ .

Since  $g \neq \emptyset$ , it contains at least one ordered pair  $(a, b)$ , and hence  $A'$  is not empty.

Let  $a \in A'$ .

Then  $\exists b \in B$  with  $(a, b) \in g$ .

Suppose  $\exists b_1, b_2 \in B$  with  $(a, b_1) \in g$  and  $(a, b_2) \in g$ .

Then  $(a, b_1) \in f$  and  $(a, b_2) \in f$ , and, since  $f$  is a function,  $b_1 = b_2$ .

Hence for all  $a \in A'$ , there exists a unique  $b \in B$  with  $(a, b) \in g$ .

Thus,  $g : A' \rightarrow B$ . ■

2. (a) Suppose  $A$  is a countably infinite set and  $b \notin A$ .

*Prove that  $A \cup \{b\}$  is countably infinite by demonstrating the necessary bijection. You need not prove it is a bijection.*

Suppose  $A$  is a countably infinite set and  $b \notin A$ .

Then there is a bijection  $f : A \rightarrow \mathbb{Z}_{>0}$ .

Define  $g : A \cup \{b\} \rightarrow \mathbb{Z}_{>0}$  by

$$g(x) = \begin{cases} 1 & \text{if } x = b; \\ f(x) + 1 & \text{if } x \neq b. \end{cases}$$

Then  $g$  is a bijection and so  $A \cup \{b\}$  is countably infinite. ■

- (b) Prove that the real intervals  $(-2, 3)$  and  $(5, 10)$  are equinumerous.

**Proof:** Let  $f : (-2, 3) \rightarrow (5, 10)$  be defined by  $f(x) = x + 7$ .

Let  $y \in (5, 10)$ .

Then  $f(y - 7) = y$  and  $y - 7 \in (-2, 3)$ .

Hence,  $f$  is onto. Suppose  $f(x_1) = f(x_2)$  for some  $x_1, x_2 \in (-2, 3)$ .

Then  $x_1 + 7 = x_2 + 7$  and so  $x_1 = x_2$ .

Hence,  $f$  is one-to-one.

Since  $f$  is one-to-one and onto,  $f$  is a bijection, so  $(-2, 3)$  and  $(5, 10)$  are equinumerous. ■

3. Let  $A$  and  $B$  be sets.

(a) Prove that  $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$ .

Let  $A$  and  $B$  be sets.

Suppose  $S \in \mathcal{P}(A) \cup \mathcal{P}(B)$ .

Then  $S \in \mathcal{P}(A)$  or  $S \in \mathcal{P}(B)$ .

Suppose  $S \in \mathcal{P}(A)$ , so  $S \subseteq A$ .

Suppose  $x \in S$ .

Then  $x \in A$  so  $x \in A \cup B$ .

Hence  $x \in S$  implies  $x \in A \cup B$ , so  $S \subseteq A \cup B$  and thus  $S \in \mathcal{P}(A \cup B)$ .

Now, suppose  $S \in \mathcal{P}(B)$ , so  $S \subseteq B$ .

Suppose  $s \in S$ .

Then  $s \in B$  so  $s \in A \cup B$ .

Hence,  $s \in S$  implies  $s \in A \cup B$ , so  $S \subseteq A \cup B$ , and thus  $S \in \mathcal{P}(A \cup B)$ .

Thus  $S \in \mathcal{P}(A) \cup \mathcal{P}(B)$  implies  $S \in \mathcal{P}(A \cup B)$ , and so  $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$ . ■

(b) Prove that there exist sets  $A$  and  $B$  such that  $\mathcal{P}(A \cup B) \not\subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$ .

Let  $A = \{1\}$  and  $B = \{2\}$ .

Then  $A \cup B = \{1, 2\}$ , so  $\{1, 2\} \in \mathcal{P}(A \cup B)$ .

Since  $2 \notin A$ ,  $\{1, 2\} \not\subseteq A$ , so  $\{1, 2\} \notin \mathcal{P}(A)$ .

Since  $1 \notin B$ ,  $\{1, 2\} \not\subseteq B$ , so  $\{1, 2\} \notin \mathcal{P}(B)$ .

Hence,  $\{1, 2\} \notin \mathcal{P}(A) \cup \mathcal{P}(B)$ .

Thus,  $S \in \mathcal{P}(A \cup B)$  does not imply  $S \in \mathcal{P}(A) \cup \mathcal{P}(B)$ .

Hence,  $\mathcal{P}(A \cup B) \not\subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$ . ■

4. Prove that, if  $n$  is an integer and  $3|n$  and  $5|n$ , then  $15|n$ .

**Proof:**

Let  $n$  be an integer and suppose  $3 | n$  and  $5 | n$ .

By Euclid's Division Theorem,  $n = 15k + r$  where  $k, r \in \mathbb{Z}$  and  $0 \leq r < 15$ .

Since  $5|n$ , we may write  $n = 5m$  for some integer  $m$ , and so

$$5m = 15 + r$$

so  $r = 5m - 15 = 5(m - 3)$ , and so  $5|r$ .

Hence,  $r = 0, 5$  or  $10$  (the only non-negative multiples of 5 less than 15).

On the other hand,  $3|n$ , so  $n = 3z$  for some integer  $z$  and so

$$3z = 15 + r$$

so  $r = 3z - 15 = 3(z - 5)$ , and so  $3|r$ .

Since 5 and 10 are not divisible by 3, we conclude that  $r = 0$  and so  $15 | n$ . ■

5. Let  $A$  be the set of all real functions  $f : \mathbb{R} \rightarrow \mathbb{R}$ .

Define a relation  $R$  on  $A$  by

$$(f, g) \in R \text{ iff } \exists k \in \mathbb{R} \text{ such that } f(x) = g(x) + k \text{ for all } x \in \mathbb{R}.$$

Prove that  $R$  is an equivalence relation.

**Proof:** Let  $A$  and  $R$  be defined as above.

Let  $f \in A$ .

Then  $f(x) = f(x) + 0$  for all  $x \in \mathbb{R}$ , so  $(f, f) \in R$ .

Thus,  $R$  is reflexive.

Suppose  $(f, g) \in A$ .

Then  $\exists k \in \mathbb{R}$  such that  $f(x) = g(x) + k$  for all  $x \in \mathbb{R}$ .

Then  $g(x) = f(x) + (-k)$  for all  $x \in \mathbb{R}$ .

Hence,  $(g, f) \in R$ , and so  $R$  is symmetric.

Suppose  $(f, g) \in A$  and  $(g, h) \in A$ .

Then  $\exists k, m \in \mathbb{R}$  such that  $f(x) = g(x) + k$  and  $g(x) = h(x) + m$  for all  $x \in \mathbb{R}$ .

Then  $f(x) = h(x) + (k + m)$  for all  $x \in \mathbb{R}$ .

Hence,  $(f, h) \in R$  and so  $R$  is transitive.

Since  $R$  is reflexive, symmetric and transitive,  $R$  is an equivalence relation. ■

6. Use induction to prove that, for all positive integers  $n$ ,

$$\sum_{j=1}^n \frac{1}{j(j+1)} = \frac{n}{n+1}.$$

**Proof:** For  $n$  a positive integer, let  $P(n)$  be the statement

$$\sum_{j=1}^n \frac{1}{j(j+1)} = \frac{n}{n+1}.$$

**Base case:** Let  $n = 1$ .

Then

$$\sum_{j=1}^1 \frac{1}{j(j+1)} = \sum_{j=1}^1 \frac{1}{j(j+1)} = \frac{1}{1(2)} = \frac{1}{2} = \frac{1}{1+1} = \frac{n}{n+1}.$$

Hence,  $P(1)$  is true.

**Induction step:** Suppose  $P(k)$  is true for some integer  $k \geq 1$ .

Then

$$\begin{aligned} \sum_{j=1}^{k+1} \frac{1}{j(j+1)} &= \sum_{j=1}^k \frac{1}{j(j+1)} + \frac{1}{(k+1)(k+2)} \\ &= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\ &= \frac{k(k+2) + 1}{(k+1)(k+2)} \\ &= \frac{k^2 + 2k + 1}{(k+1)(k+2)} \\ &= \frac{(k+1)^2}{(k+1)(k+2)} \\ &= \frac{k+1}{k+2}. \end{aligned}$$

Hence,  $P(k+1)$  is true.

Thus,  $P(k)$  implies  $P(k+1)$ , and, since  $P(1)$  is true, by induction  $P(n)$  is true for all  $n \geq 1$ . ■

## Elementary Properties of the Integers (EPIs)

Suppose  $a, b, c$ , and  $d$  are integers.

- **Closure:**  $a + b$  and  $ab$  are integers.
- **Substitution of Equals:** If  $a = b$ , then  $a + c = b + c$  and  $ac = bc$ .
- **Commutativity:**  $a + b = b + a$  and  $ab = ba$ .
- **Associativity:**  $(a + b) + c = a + (b + c)$  and  $(ab)c = a(bc)$ .
- **The Distributive Law:**  $a(b + c) = ab + ac$
- **Identities:**  $a + 0 = 0 + a = a$  and  $a \cdot 1 = 1 \cdot a = a$ .  
 $0$  is called the additive identity.  
 $1$  is called the multiplicative identity.
- **Additive Inverses:** There exists an integer  $-a$  such that  $a + (-a) = (-a) + a = 0$ .
- **Trichotomy:** Exactly one of the following is true:  
 $a > 0$ ,  $-a > 0$ , or  $a = 0$ .
- **The Well-Ordering Principle:** Every non-empty set of positive integers contains a smallest element.

1.  $a \cdot 0 = 0$
2. If  $a + c = b + c$ , then  $a = b$ .
3.  $-a = (-1) \cdot a$
4.  $(-a) \cdot b = -(ab)$
5.  $(-a) \cdot (-b) = ab$
6. If  $ab = 0$ , then  $a = 0$  or  $b = 0$ .
7. If  $a \leq b$  and  $b \leq a$ , then  $a = b$ .
8. If  $a < b$  and  $b < c$ , then  $a < c$ .
9. If  $a < b$ , then  $a + c < b + c$ .
10. If  $a < b$  and  $0 < c$ , then  $ac < bc$ .
11. If  $a < b$  and  $c < 0$ , then  $bc < ac$ .
12. If  $a < b$  and  $c < d$ , then  $a + c < b + d$ .
13. If  $0 \leq a < b$  and  $0 \leq c < d$ , then  $ac < bd$ .
14. If  $a < b$ , then  $-b < -a$ .
15.  $0 \leq a^2$ , where  $a^2 = a \cdot a$ .
16. If  $ab = 1$ , then either  $a = b = 1$  or  $a = b = -1$ .

### Sets

$A \subseteq B$  iff  $x \in A$  implies  $x \in B$

$A = B$  iff  $A \subseteq B$  and  $B \subseteq A$

$x \in A \cup B$  iff  $x \in A$  or  $x \in B$

$x \in A \cap B$  iff  $x \in A$  and  $x \in B$

$x \in A \setminus B$  iff  $x \in A$  and  $x \notin B$

$\mathcal{P}(A)$  is the set of all subsets of a set  $A$

For a family  $\mathcal{F}$ :

$$\bigcup \mathcal{F} = \{x : \exists S \in \mathcal{F} \text{ such that } x \in S\}$$

$$\bigcap \mathcal{F} = \{x : \forall S \in \mathcal{F}, x \in S\}$$

NOTE: Properties 8-14 hold if each  $<$  is replaced with  $\leq$ .

**Euclid's Division Theorem:** For all positive integers  $m$  and all integers  $n$ , there exist unique integers  $q$  and  $r$ , with  $0 \leq r < m$  such that  $n = qm + r$ .

**Bijections have inverse functions Theorem:** Let  $f : A \rightarrow B$  be a bijection, where  $A$  and  $B$  are sets. Then  $f^{-1} : B \rightarrow A$  and is a bijection.

**Inverse composition Theorem:** A function  $f : A \rightarrow B$  is a bijection iff there is a function  $g : B \rightarrow A$  such that  $f \circ g = i_B$  and  $g \circ f = i_A$ .