

Math 300 D - Autumn 2024

Final Exam

December 10, 2024

Solutions

1. *By considering the situation modulo 8, show that there are infinitely many positive integers that are not expressible in the form $a^2 + 2b^2$ where a and b are non-negative integers.*

Every integer is congruent to an element of

$$\{0, 1, 2, 3, 4, 5, 6, 7\}$$

modulo 8.

Here, we calculate squares and twice squares of these values modulo 8:

m	m^2	$2m^2$
0	0	0
1	1	2
2	4	0
3	$9 \equiv 1$	2
4	$16 \equiv 0$	0
5	$25 \equiv 1$	2
6	$36 \equiv 4$	0
7	$49 \equiv 1$	2

So, we see that, for any integer a , $a^2 \equiv 0, 1$ or $4 \pmod{8}$.

Also, for any integer b , $2b^2 \equiv 0$ or $2 \pmod{8}$.

Thus, the possible values of $a^2 + 2b^2$ modulo 8 are shown in the following table:

$+$	0	2
0	0	2
1	1	3
4	4	6

So we see that, for any integers a and b , $a^2 + 2b^2$ cannot be congruent to 5 or 7 modulo 8.

Since $8k + 5$ is a positive integer congruent to 5 modulo 8 for every positive integer k , and none of them can be represented in the form $a^2 + 2b^2$, we conclude that there are infinitely many positive integers that cannot be represented in the form $a^2 + 2b^2$. ■

2. Suppose $f : A \rightarrow C$ and $g : B \rightarrow C$.

Prove that if $A \cap B = \emptyset$, then $f \cup g : A \cup B \rightarrow C$.

Proof: Suppose $f : A \rightarrow C$ and $g : B \rightarrow C$, and suppose $A \cap B = \emptyset$.

Then $f \subseteq A \times C$ and $g \subseteq B \times C$.

Suppose $(a, b) \in f \cup g$.

Then $(a, b) \in A \times C$ or $(a, b) \in B \times C$.

So $a \in A$ or $a \in B$, that is, $a \in A \cup B$.

Also, $b \in C$.

Hence, $(a, b) \in (A \cup B) \times C$.

Thus, $(a, b) \in f \cup g$ implies $(a, b) \in (A \cup B) \times C$, so $f \cup g \subseteq (A \cup B) \times C$.

Suppose $x \in A \cup B$.

Then $x \in A$ or $x \in B$.

If $x \in A$, then $(x, f(x)) \in f$.

If $x \in B$, then $(x, g(x)) \in g$.

Hence, $(x, y) \in f \cup g$ for some y .

Now, suppose $(x, y_1) \in f \cup g$ and $(x, y_2) \in f \cup g$ with $y_1 \neq y_2$.

The fact that f is a function would be contradicted if $(x, y_1) \in f$ and $(x, y_2) \in f$.

The fact that g is a function would be contradicted if $(x, y_1) \in g$ and $(x, y_2) \in g$.

The fact that $A \cap B = \emptyset$ would be contradicted if $(x, y_1) \in f$ and $(x, y_2) \in g$ (or the other way around).

Thus, for all $x \in A \cup B$, there exists a unique $y \in C$ such that $(x, y) \in f \cup g$.

Therefore,

$$f \cup g : A \cup B \rightarrow C. \quad \blacksquare$$

3. Let $c \in \mathbb{R}$ and define a relation $R \subseteq \mathbb{R} \times \mathbb{R}$ by

$$(a, b) \in R \text{ iff } |a - b| \leq c.$$

For what c is R an equivalence relation? Prove that your answer is correct.

Let $c \in \mathbb{R}$ and define R as above.

Suppose $c < 0$.

Since $|x| \geq 0$ for any real number x , no $a, b \in \mathbb{R}$ could satisfy $|a - b| \leq c$.

As a result, R is not reflexive ($(a, a) \notin R$ for any $a \in \mathbb{R}$).

Hence, in order for R to be an equivalence relation, we need $c \geq 0$.

Suppose $c > 0$.

Let $x = 0, y = c$ and $z = 2c$.

Then $|x - y| = c \leq c$ and $|z - x| = c \leq c$.

Hence, $(x, y) \in R$ and $(y, z) \in R$.

However, $|x - z| = 2c > c$, so $(x, y) \in R$ and $(y, z) \in R$ does not imply $(x, z) \in R$.

Hence, R is not transitive.

Thus, in order for R to be an equivalence relation, we need $c = 0$.

Now, suppose $c = 0$.

Then $|a - b| \leq c$ is equivalent to $a = b$: $|a - b| \leq 0$ implies $|a - b| = 0$ which implies $a - b = 0$ which implies $a = b$, and $a = b$ implies $|a - b| = 0$.

Since equality is reflexive, symmetric and transitive, R is an equivalence relation.

Thus, R is an equivalence relation if and only if $c = 0$. ■

4. Use induction to prove that for all positive integers n

$$\sum_{i=1}^n i(i!) = (n+1)! - 1.$$

Proof:

Let $P(n) = " \sum_{i=1}^n i(i!) = (n+1)! - 1 "$ for any positive integer n .

Base case: Let $n = 1$.

Then

$$\sum_{i=1}^n i(i!) = \sum_{i=1}^1 i(i!) = 1(1!) = 1 = 2! - 1 = (n+1)! - 1.$$

So $P(1)$ is true.

Induction step: Suppose $P(n)$ is true for some $n = k \in \mathbb{Z}_{>0}$.

Then

$$\sum_{i=1}^k i(i!) = (k+1)! - 1.$$

Then

$$\begin{aligned} \sum_{i=1}^{k+1} i(i!) &= (k+1)! - 1 + (k+1)((k+1)!) \\ &= (k+1)!(1 + k+1) - 1 \\ &= (k+2)((k+1)!) - 1 \\ &= (k+2)! - 1. \end{aligned}$$

Hence, $P(k+1)$ is true.

Thus, for $n \in \mathbb{Z}_{>0}$, $P(n)$ implies $P(n+1)$, and since $P(1)$ is true, by induction $P(n)$ is true for all $n \in \mathbb{Z}_{>0}$. ■

5. Let $\mathcal{F}$ and $\mathcal{G}$ be families.

(a) Prove that $\bigcup \mathcal{F} \setminus \bigcup \mathcal{G} \subseteq \bigcup (\mathcal{F} \setminus \mathcal{G})$.

Suppose $x \in \bigcup \mathcal{F} \setminus \bigcup \mathcal{G}$.

Then $x \in \bigcup \mathcal{F}$ and $x \notin \bigcup \mathcal{G}$.

Hence $\exists S \in \mathcal{F}$ such that $x \in S$.

Also, $S \notin \mathcal{G}$ since $x \notin \bigcup \mathcal{G}$.

Thus $S \in \mathcal{F} \setminus \mathcal{G}$ and so $x \in \bigcup (\mathcal{F} \setminus \mathcal{G})$.

So, $x \in \bigcup \mathcal{F} \setminus \bigcup \mathcal{G}$ implies $x \in \bigcup (\mathcal{F} \setminus \mathcal{G})$.

Therefore $\bigcup \mathcal{F} \setminus \bigcup \mathcal{G} \subseteq \bigcup (\mathcal{F} \setminus \mathcal{G})$. ■

(b) Give an example of $\mathcal{F}$ and $\mathcal{G}$ such that $\bigcup (\mathcal{F} \setminus \mathcal{G}) \not\subseteq \bigcup \mathcal{F} \setminus \bigcup \mathcal{G}$.

Let $\mathcal{F} = \{\{x\}\}$.

Let $\mathcal{G} = \{\{x, y\}\}$.

Then $\mathcal{F} \setminus \mathcal{G} = \mathcal{F}$, so $\bigcup (\mathcal{F} \setminus \mathcal{G}) = \bigcup \mathcal{F} = \{x\}$.

Also, $\bigcup \mathcal{F} = \{x\}$ and $\bigcup \mathcal{G} = \{x, y\}$, so $\bigcup \mathcal{F} \setminus \bigcup \mathcal{G} = \emptyset$.

Since $x \in \bigcup (\mathcal{F} \setminus \mathcal{G})$ and $x \notin \emptyset$,

$$\bigcup (\mathcal{F} \setminus \mathcal{G}) \not\subseteq \bigcup \mathcal{F} \setminus \bigcup \mathcal{G}.$$

6. (a) Give a bijection between the real number intervals $(-2, 6)$ and $(3, 17)$. You need not prove that it is a bijection.

One bijection is given by $f(x) = \frac{7}{4}(x + 2) + 3$.

- (b) Give a bijection from $\mathbb{Z}$ to $\mathbb{Z} \setminus \{-3, 2\}$. You need not prove that it is a bijection.

The following works:

$$g(x) = \begin{cases} x & \text{if } x \leq -4; \\ x + 1 & \text{if } -3 \leq x \leq 0; \\ x + 2 & \text{if } x \geq 1. \end{cases}$$

Elementary Properties of the Integers (EPIs)

Suppose a, b, c , and d are integers.

- **Closure:** $a + b$ and ab are integers.
- **Substitution of Equals:** If $a = b$, then $a + c = b + c$ and $ac = bc$.
- **Commutativity:** $a + b = b + a$ and $ab = ba$.
- **Associativity:** $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$.
- **The Distributive Law:** $a(b + c) = ab + ac$
- **Identities:** $a + 0 = 0 + a = a$ and $a \cdot 1 = 1 \cdot a = a$.
0 is called the additive identity.
1 is called the multiplicative identity.
- **Additive Inverses:** There exists an integer $-a$ such that $a + (-a) = (-a) + a = 0$.
- **Trichotomy:** Exactly one of the following is true:
 $a > 0$, $-a > 0$, or $a = 0$.
- **The Well-Ordering Principle:** Every non-empty set of positive integers contains a smallest element.

1. $a \cdot 0 = 0$
2. If $a + c = b + c$, then $a = b$.
3. $-a = (-1) \cdot a$
4. $(-a) \cdot b = -(ab)$
5. $(-a) \cdot (-b) = ab$
6. If $ab = 0$, then $a = 0$ or $b = 0$.
7. If $a \leq b$ and $b \leq a$, then $a = b$.
8. If $a < b$ and $b < c$, then $a < c$.
9. If $a < b$, then $a + c < b + c$.
10. If $a < b$ and $0 < c$, then $ac < bc$.
11. If $a < b$ and $c < 0$, then $bc < ac$.
12. If $a < b$ and $c < d$, then $a + c < b + d$.
13. If $0 \leq a < b$ and $0 \leq c < d$, then $ac < bd$.
14. If $a < b$, then $-b < -a$.
15. $0 \leq a^2$, where $a^2 = a \cdot a$.
16. If $ab = 1$, then either $a = b = 1$ or $a = b = -1$.

Sets

$A \subseteq B$ iff $x \in A$ implies $x \in B$

$A = B$ iff $A \subseteq B$ and $B \subseteq A$

$x \in A \cup B$ iff $x \in A$ or $x \in B$

$x \in A \cap B$ iff $x \in A$ and $x \in B$

$x \in A \setminus B$ iff $x \in A$ and $x \notin B$

NOTE: Properties 8-14 hold if each $<$ is replaced with $\leq$.

Two theorems for reference:

Theorem DAS (Divisors are Smaller):

Let a and b be positive integers. Then $a|b$ implies $a \leq b$.

Euclid's Division Theorem:

Let m be a positive integer. For all $n \in \mathbb{Z}$, there exist integers k and r such that

$$n = mk + r$$

with $0 \leq r < m$.