

Math 300 C - Autumn 2024
Final Exam
December 9, 2024

Name: _____

Student ID no. : _____

Signature: _____

- You have 110 minutes to complete the exam.
- When time is called, you must stop writing immediately.

1. By considering the situation modulo 7, show that there are infinitely many positive integers that are not expressible in the form $2^a + b^3$, where a and b are non-negative integers.

2. Let A , B and C be non-empty sets, and suppose $f : A \rightarrow C$ and $g : B \rightarrow C$.

Let $h = f \cup g$. Then $h \subseteq (A \cup B) \times C$.

Show that $h : A \cup B \rightarrow C$ iff $f(x) = g(x)$ for all $x \in A \cap B$.

3. Let

$$f : \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$$

be defined by

$$f(x) = \{z \in \mathbb{R} : x + z \in \mathbb{Z}\}.$$

Show that f is not one-to-one and not onto.

4. (a) Is the inverse of an equivalence relation necessarily an equivalence relation? Prove you are correct.

(b) Give an example of an equivalence relation on the set $A = \{a, b, c\}$ that is not the identity function on A . You need not prove it is an equivalence relation.

5. Prove the following two theorems.

Let A and B be sets. Then:

(a) If $S \in \mathcal{P}(A \setminus B)$, and $S \neq \emptyset$, then $S \in \mathcal{P}(A) \setminus \mathcal{P}(B)$.

(b) $\mathcal{P}(A) \setminus \mathcal{P}(B)$ need not be a subset of $\mathcal{P}(A \setminus B)$.

6. Use induction to prove that, for all positive integers n ,

$$16 \mid 5^n + 12n - 1.$$

Elementary Properties of the Integers (EPIs)

Suppose a, b, c , and d are integers.

- **Closure:** $a + b$ and ab are integers.
- **Substitution of Equals:** If $a = b$, then $a + c = b + c$ and $ac = bc$.
- **Commutativity:** $a + b = b + a$ and $ab = ba$.
- **Associativity:** $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$.
- **The Distributive Law:** $a(b + c) = ab + ac$
- **Identities:** $a + 0 = 0 + a = a$ and $a \cdot 1 = 1 \cdot a = a$.
 0 is called the additive identity.
 1 is called the multiplicative identity.
- **Additive Inverses:** There exists an integer $-a$ such that $a + (-a) = (-a) + a = 0$.
- **Trichotomy:** Exactly one of the following is true:
 $a > 0$, $-a > 0$, or $a = 0$.
- **The Well-Ordering Principle:** Every non-empty set of positive integers contains a smallest element.

1. $a \cdot 0 = 0$
2. If $a + c = b + c$, then $a = b$.
3. $-a = (-1) \cdot a$
4. $(-a) \cdot b = -(ab)$
5. $(-a) \cdot (-b) = ab$
6. If $ab = 0$, then $a = 0$ or $b = 0$.
7. If $a \leq b$ and $b \leq a$, then $a = b$.
8. If $a < b$ and $b < c$, then $a < c$.
9. If $a < b$, then $a + c < b + c$.
10. If $a < b$ and $0 < c$, then $ac < bc$.
11. If $a < b$ and $c < 0$, then $bc < ac$.
12. If $a < b$ and $c < d$, then $a + c < b + d$.
13. If $0 \leq a < b$ and $0 \leq c < d$, then $ac < bd$.
14. If $a < b$, then $-b < -a$.
15. $0 \leq a^2$, where $a^2 = a \cdot a$.
16. If $ab = 1$, then either $a = b = 1$ or $a = b = -1$.

Sets

$A \subseteq B$ iff $x \in A$ implies $x \in B$

$A = B$ iff $A \subseteq B$ and $B \subseteq A$

$x \in A \cup B$ iff $x \in A$ or $x \in B$

$x \in A \cap B$ iff $x \in A$ and $x \in B$

$x \in A \setminus B$ iff $x \in A$ and $x \notin B$

NOTE: Properties 8-14 hold if each $<$ is replaced with $\leq$.

Two theorems for reference:

Theorem DAS (Divisors are Smaller):

Let a and b be positive integers. Then $a|b$ implies $a \leq b$.

Euclid's Division Theorem:

Let m be a positive integer. For all $n \in \mathbb{Z}$, there exist integers k and r such that

$$n = mk + r$$

with $0 \leq r < m$.