

Math 300 D - Autumn 2014
Final Exam
December 9, 2014
Solutions

1. Let $\mathcal{F}$ and $\mathcal{G}$ be families of sets. Prove that $(\cap \mathcal{F}) \cap (\cap \mathcal{G}) = \cap(\mathcal{F} \cup \mathcal{G})$.

Proof: Let $\mathcal{F}$ and $\mathcal{G}$ be families of sets.

Suppose $x \in \cap \mathcal{F} \cap \cap \mathcal{G}$.

Then $x \in \cap \mathcal{F}$ and $x \in \cap \mathcal{G}$.

Suppose $M \in \mathcal{F}$. Then $x \in M$.

Suppose $N \in \mathcal{G}$. Then $x \in N$.

Suppose $P \in \mathcal{F} \cup \mathcal{G}$. Then $P \in \mathcal{F}$ or $P \in \mathcal{G}$. Hence, $x \in P$.

Thus, x is an element of every set in $\mathcal{F} \cup \mathcal{G}$, so $x \in \cap(\mathcal{F} \cup \mathcal{G})$.

Hence, $(\cap \mathcal{F}) \cap (\cap \mathcal{G}) \subseteq \cap(\mathcal{F} \cup \mathcal{G})$.

Now, suppose $x \in \cap(\mathcal{F} \cup \mathcal{G})$.

Suppose $M \in \mathcal{F} \cup \mathcal{G}$. Then $x \in M$.

Suppose $N \in \mathcal{F}$. Then $N \in \mathcal{F} \cup \mathcal{G}$, and so $x \in N$.

Hence, x is in every set in $\mathcal{F}$, and so $x \in \cap \mathcal{F}$.

Suppose $P \in \mathcal{G}$. Then $P \in \mathcal{F} \cup \mathcal{G}$, and so $x \in P$.

Hence, x is in every set in $\mathcal{G}$, and so $x \in \cap \mathcal{G}$.

Thus, $x \in (\cap \mathcal{F}) \cap (\cap \mathcal{G})$.

Therefore $\cap(\mathcal{F} \cup \mathcal{G}) \subseteq (\cap \mathcal{F}) \cap (\cap \mathcal{G})$, and so $(\cap \mathcal{F}) \cap (\cap \mathcal{G}) = \cap(\mathcal{F} \cup \mathcal{G})$. ■

2. Let a and b be integers. Prove that $a(b + a + 1)$ is odd iff a and b are both odd.

Proof: Let a and b be integers.

Suppose a is even.

Then $a = 2k$ for some $k \in \mathbb{Z}$.

Then $a(b + a + 1) = 2k(b + a + 1)$, and since a, b , and k are integers, $k(b + a + 1)$ is an integer.

Hence, $2 \mid a(b + a + 1)$, i.e., $a(b + a + 1)$ is even.

Suppose a is odd and b is even.

Then $a = 2k + 1$ and $b = 2m$ for some integers k and m .

Then $a(b + a + 1) = a(2k + 2m + 1 + 1) = a(2k + 2m + 2) = 2a(k + m + 1)$. Since a, k , and m are integers, $a(k + m + 1)$ is an integer, and so $2 \mid a(b + a + 1)$, i.e., $a(b + a + 1)$ is even.

Suppose a is odd and b is odd.

Then $a = 2k + 1$ and $b = 2m + 1$ for some integers k and m .

Then

$$\begin{aligned} a(b + a + 1) &= (2k + 1)(2k + 2m + 3) \\ &= 4k^2 + 4mk + 8k + 2m + 3 \\ &= 2(2k^2 + 4mk + 4k + m + 1) + 1. \end{aligned}$$

Since k and m are integers, $2k^2 + 4mk + 4k + m + 1$ is an integer, and so $a(b + a + 1)$ is odd.

Thus, $a(b + a + 1)$ is odd iff a and b are both odd. ■

3. Let A, B, C , and D be sets. Suppose $A \cap C = B \cap D = \emptyset$. Suppose $A \sim B$ and $C \sim D$. Prove that $A \cup C \sim B \cup D$.

Proof: Let A, B, C , and D be sets. Suppose $A \cap C = B \cap D = \emptyset$. Suppose $A \sim B$ and $C \sim D$. Then there exist bijections $f : A \rightarrow B$ and $g : C \rightarrow D$.

Define a function $h : A \cup C \rightarrow B \cup D$ by

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in C \end{cases}$$

Since $A \cap C = \emptyset$, the function is well-defined: given an x in $A \cup C$, x is either in A or C , but not both.

Suppose $h(x_1) = h(x_2)$ for some x_1 and x_2 in $A \cup C$.

Suppose $x_1 \in A$, and $x_2 \in A$. Then $h(x_1) = f(x_1) = h(x_2) = f(x_2)$, and so $x_1 = x_2$ since f is injective.

Suppose $x_1 \in C$, and $x_2 \in C$. Then $h(x_1) = g(x_1) = h(x_2) = g(x_2)$, and so $x_1 = x_2$ since g is injective.

Suppose x_1 and x_2 are not both in A or both in C . Without loss of generality, suppose $x_1 \in A$ and $x_2 \in C$. Then $h(x_1) = h(x_2)$ implies that $f(x_1) = g(x_2)$. But $f(x_1) \in B$ and $g(x_2) \in D$, so $B \cap D \neq \emptyset$. This is a contradiction to our assumption that $B \cap D = \emptyset$. Hence, x_1 and x_2 must both be in A or both be in C .

Thus, $x_1 = x_2$, and so h is one-to-one.

Suppose $r \in B \cup D$.

Then $r \in B$ or $r \in D$.

Suppose $r \in B$. Then, since f is surjective, there exists an $a \in A$ such that $f(a) = r$.

Also, $a \in A \cup C$, and $h(a) = r$.

Suppose $r \in D$. Then, since g is surjective, there exists a $c \in C$ such that $g(c) = r$.

Also, $c \in A \cup C$ and $h(c) = r$.

Hence, there exists $x \in A \cup C$ with $h(x) = r$, so h is surjective.

Thus, h is a bijection, and so $A \cup C \sim B \cup D$. ■

4. (a) Give an example of a function $f : \mathbb{Z}_{>0} \rightarrow \mathbb{Z}_{>0}$ such that f is one-to-one, but not onto (i.e., f is injective but not surjective). Prove that f is one-to-one and not onto.

Note: There are many examples you could give. If you are using this exam solution to study, I recommend creating a few, as different from each other as possible.

One example is $f(x) = x + 1$.

We note that since $x > 0$, $x + 1 > 1$ and so there is no a such that $f(a) = 1$. So f is not onto.

Suppose $f(x_1) = f(x_2)$. Then $x_1 + 1 = x_2 + 1$, and so $x_1 = x_2$. So f is one-to-one. ■

- (b) Give an example of a function $g : \mathbb{Z}_{>0} \rightarrow \mathbb{Z}_{>0}$ such that g is onto, but not one-to-one (i.e., g is surjective, but not injective). Prove that g is onto and not one-to-one.

Note: There are many examples you could give. If you are using this exam solution to study, I recommend creating a few, as different from each other as possible.

Here is one example. Let

$$g(x) = \begin{cases} x & \text{if } x \text{ is odd,} \\ x/2 & \text{if } x \text{ is even.} \end{cases}$$

Let a be a positive integer.

Suppose a is odd. Then $g(a) = a$.

Suppose a is even. Then $2a \in \mathbb{Z}_{>0}$ and $g(2a) = a$.

Thus, g is onto.

On the other hand, $g(6) = g(3) = 3$, so g is not one-to-one. ■

5. Use induction to prove that $n! > n^2$ for all integers $n \geq 4$.

Proof: Define $P(n)$ to be the statement " $n! > n^2$ ".

Base case: Let $n = 4$. Then $n! = 24 > 16 = n^2$, so $P(4)$ is true.

Induction step: Suppose $P(n)$ is true for some $n = x \geq 4$.

Then $x! > x^2$. We want to show that $(x + 1)! > (x + 1)^2$.

We have

$$\begin{aligned} (x + 1)! &= (x + 1) \cdot x! \\ &> (x + 1)x^2 \quad (\text{by our induction hypothesis}) \\ &= x^3 + x^2 \\ &= x \cdot x^2 + x^2 \\ &= (x - 1) \cdot x^2 + x^2 + x^2 \\ &\geq 3x^2 + 4x + 16 \\ &> x^2 + 2x + 1 \\ &= (x + 1)^2. \end{aligned}$$

Thus, $(x + 1)! > x^2$ and so $P(x + 1)$ is true.

Thus, $P(x)$ implies $P(x + 1)$, and since $P(4)$ is true, by induction $P(n)$ is true for all integers $n \geq 4$. ■

6. Let R be the relation defined on the real numbers, $\mathbb{R}$, by

$$(x, y) \in R \Leftrightarrow \text{there exist positive integers } n \text{ and } m \text{ such that } x^n = y^m.$$

Prove that R is an equivalence relation.

Proof: Let $x \in \mathbb{R}$. Then $x^1 = x^1$, and so $(x, x) \in R$.

Hence, R is reflexive.

Suppose $(x, y) \in R$. Then $x^m = y^n$ for some integers m and n .

Then, $y^n = x^m$, and so $(y, x) \in R$.

Hence R is symmetric.

Suppose $(x, y) \in R$ and $(y, z) \in R$.

Then $x^m = y^n$ and $y^r = z^s$ for positive integers m, n, r , and s .

Then $x^{rm} = y^{rn}$ and $y^{rn} = z^{sn}$, so $x^{rm} = z^{sn}$.

Since rm and sn are positive integers, we conclude that $(x, z) \in R$.

Hence, R is transitive.

Since R is reflexive, symmetric and transitive, R is an equivalence relation. ■