How to extract derivative values from Taylor series

Since the Taylor series of \( f \) based at \( x = b \) is

\[
\sum_{n=0}^{\infty} \frac{f^{(n)}(b)}{n!} (x - b)^n,
\]

we may think of the Taylor series as an encoding of all of the derivatives of \( f \) at \( x = b \): that information is in there.

As a result, if we know the Taylor series for a function, we can extract from it any derivative of the function at \( b \).

Here are a few examples.

**Example.** Let \( f(x) = x^2 e^{3x} \). Find \( f^{11}(0) \).

The Taylor series for \( e^x \) based at \( b = 0 \) is

\[
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!},
\]

so we have

\[
e^{3x} = \sum_{n=0}^{\infty} \frac{(3x)^n}{n!}
\]

and

\[
x^2 e^{3x} = \sum_{n=0}^{\infty} \frac{3^n x^{n+2}}{n!} = \sum_{m=2}^{\infty} \frac{3^{m-2}}{(m-2)!} x^m.
\]

We can see that, for \( m \geq 2 \) the coefficient on \( x^m \) is

\[
\frac{3^{m-2}}{(m-2)!}.
\]

On the other hand, this is the Taylor series for \( f(x) \) based at \( b = 0 \), and so the coefficient on \( x^m \) is equal to

\[
\frac{f^{(m)}(0)}{m!}.
\]

Equating these two, we have

\[
\frac{f^{(m)}(0)}{m!} = \frac{3^{m-2}}{(m-2)!}
\]

and we can say

\[
f^{(m)}(0) = 3^{m-2} \frac{m!}{(m-2)!} = 3^{m-2} m(m-1).
\]

Thus, taking \( m = 11 \), we have

\[
f^{(11)}(0) = 3^9 (11)(10) = 2165130.
\]
**Example.** Let $f(x) = \cos x^2$. Find $f^{(88)}(0)$.

We know the Taylor series for $\cos x$ based at $b = 0$ is

$$
\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}
$$

By substitution, we then quickly find

$$
\cos x^2 = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}
$$

and we may simplify this to

$$
\cos x^2 = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{(2n)!}
$$

Now, with $f(x) = \cos x^2$, and $b=0$, we have

$$
\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{(2n)!} = \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!} x^j.
$$

Here I rewrote the general Taylor series based at zero with then index $j$ to help our thinking.

From this, we can see that if $j$ is not a multiple of four, then $f^{(j)}(0)=0$, since the only powers of $x$ which appear in the Taylor series are multiples of four. If $j$ is a multiple of four, say $j = 4n$, then

$$
\frac{f^{(j)}(0)}{j!} = \frac{(-1)^n}{(2n)!}
$$

by matching up the coefficients: the coefficient on each power of $x$ in the left- and right-hand expressions must be the same.

Thus, we can say

$$
f^{(j)}(0) = (-1)^n \frac{j!}{(2n)!} = (-1)^{j/4} \frac{j!}{(j/2)!}.
$$

Finally, we may conclude that

$$
f^{(88)}(0) = (-1)^{44} \frac{88!}{44!} \approx 6.9776 \times 10^7.
$$