How to extract derivative values from Taylor series

Since the Taylor series of f based at $x = b$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(b)}{n!}(x-b)^n,$$

we may think of the Taylor series as an *encoding* of all of the derivatives of f at $x = b$: that information is *in there*.

As a result, if we know the Taylor series for a function, we can extract from it any derivative of the function at b.

Here are a few examples.

Example. Let $f(x) = x^2 e^{3x}$. Find $f^{11}(0)$.

The Taylor series for e^x based at $b = 0$ is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

so we have

$$e^{3x} = \sum_{n=0}^{\infty} \frac{(3x)^n}{n!}$$

and

$$x^2 e^{3x} = \sum_{n=0}^{\infty} \frac{3^n x^{n+2}}{n!} = \sum_{m=2}^{\infty} \frac{3^{m-2}}{(m-2)!} x^m.$$

We can see that, for $m \geq 2$ the coefficient on x^m is

$$\frac{3^{m-2}}{(m-2)!}.$$

On the other hand, this is the Taylor series for $f(x)$ based at $b = 0$, and so the coefficient on x^m is equal to

$$\frac{f^{(m)}(0)}{m!}.$$

Equating these two, we have

$$\frac{f^{(m)}(0)}{m!} = \frac{3^{m-2}}{(m-2)!}$$

and we can say

$$f^{(m)}(0) = 3^{m-2} \frac{m!}{(m-2)!} = 3^{m-2} m(m-1).$$

Thus, taking $m = 11$, we have

$$f^{(11)}(0) = 3^9(11)(10) = 2165130.$$

Example. Let $f(x) = \cos x^2$. Find $f^{(88)}(0)$.

We know the Taylor series for $\cos x$ based at $b = 0$ is

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

By substitution, we then quickly find

$$\cos x^2 = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n}}{(2n)!}$$

and we may simplify this to

$$\cos x^2 = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{(2n)!}$$

Now, with $f(x) = \cos x^2$, and b=0, we have

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{(2n)!} = \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!}(x)^j.$$

Here I rewrote the general Taylor series based at zero with then index j to help our thinking.

From this, we can see that if j is not a multiple of four, then $f^j(0)$=0, since the only powers of x which appear in the Taylor series are multiples of four. If j is a multiple of four, say $j = 4n$, then

$$\frac{f^{(j)}(0)}{j!} = \frac{(-1)^n}{(2n)!}$$

by matching up the coefficients: the coefficient on each power of x in the left- and right-hand expressions must be the same.

Thus, we can say

$$f^{(j)}(0) = (-1)^n \frac{j!}{(2n)!} = (-1)^{j/4} \frac{j!}{(j/2)!}.$$

Finally, we may conclude that

$$f^{(88)}(0) = (-1)^{44} \frac{88!}{44!} \approx 6.9776 \times 10^7 9.$$