Quadric Surfaces

In 3-dimensional space, we may consider quadratic equations in three variables x, y, and z:

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz + gx + hy + iz + j = 0$$

Such an equation defines a surface in 3D. *Quadric surfaces* are the surfaces whose equations can be, through a series of rotations and translations, put into quadratic polynomial equations of the form

$$\pm\frac{x^\alpha}{a^2} \pm \frac{y^\beta}{b^2} \pm \frac{z^\gamma}{c^2} = k \tag{1}$$

which are quadratic in at least two variables. That is, α, β and γ are all either 1 or 2, and at least two are equal to 2.

There are six distinct types of quadric surfaces, arising from different forms of equation (1).

1. Ellipsoids

   The **ellipsoid** is the surface given by equations of the form

   $$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = k$$

   for positive k.

   The cross-sections are all ellipses.

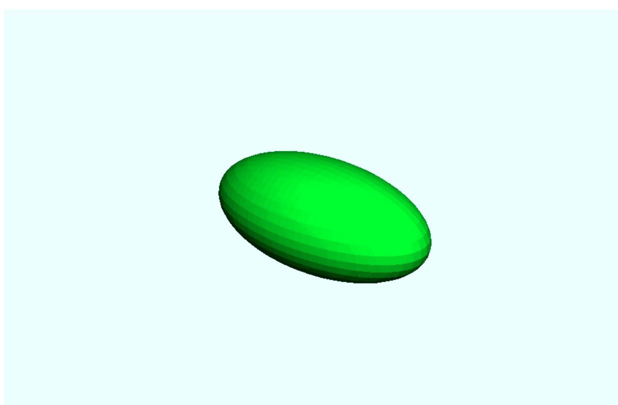


Figure 1: ellipsoid

2. Elliptic paraboloids

   The **elliptic paraboloid** is the surface given by equations of the form

   $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z}{c} = 0.$$

   Cross-sections parallel to the xy-plane are ellipses, while those parallel to the xz- and yz-planes are parabolas.

   Note that the origin satisfies this equation.

3. Hyperbolic paraboloid

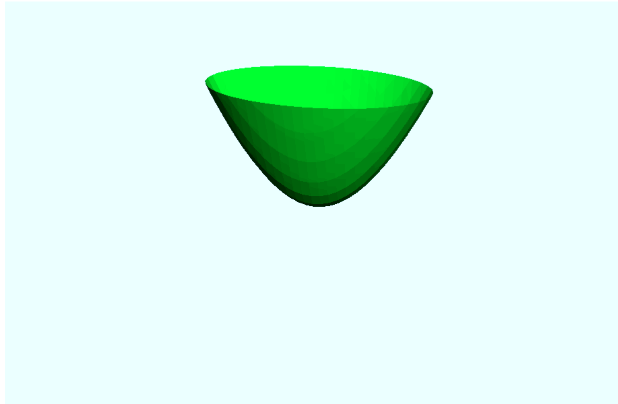


Figure 2: elliptic paraboloid

The **hyperbolic paraboloid** is the surface given by equations of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z}{c} = 0.$$

Cross-sections parallel to the xy-plane are hyperbolas, while those parallel to the xz- and yz-planes are parabolas.

This curve has a shape similar to a saddle.

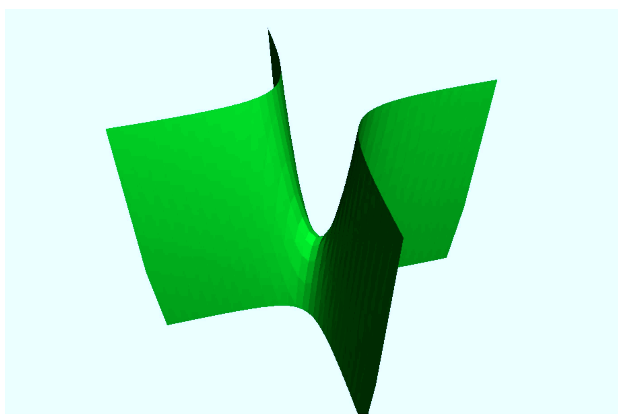


Figure 3: hyperbolic paraboloid

4. Cones, hyperboloids of one sheet and hyperboloids of two sheets

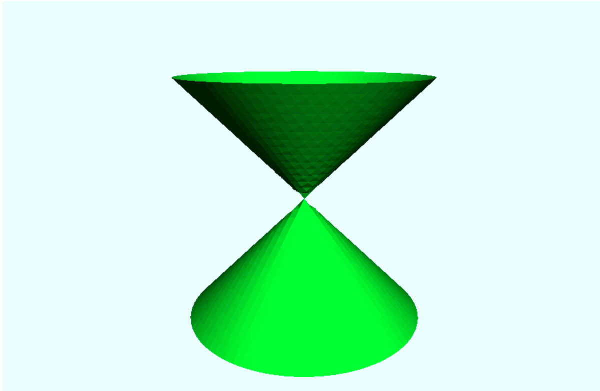
   These surfaces all result from equations of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = k$$

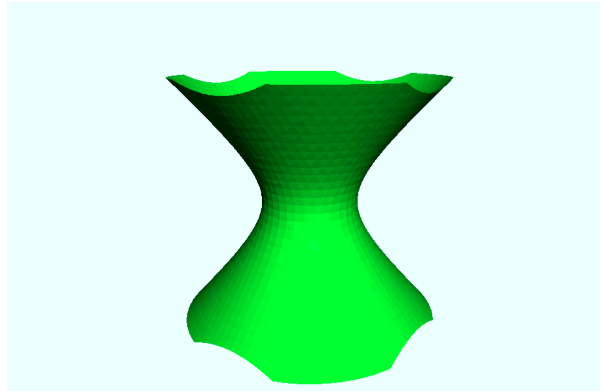
where k is a real constant. We can observe that for all such surfaces, cross-sections parallel to the xy-plane are ellipses, while cross-sections parallel to the yz-plane or the xz-plane are hyperbolas (or degenerate hyperbolas: a pair of intersecting lines).

In the case $k = 0$, the surface is a **cone**. We can observe that the surface contains the origin, and the intersection of the surface with, for instance, the yz-plane is the degenerate hyperbola

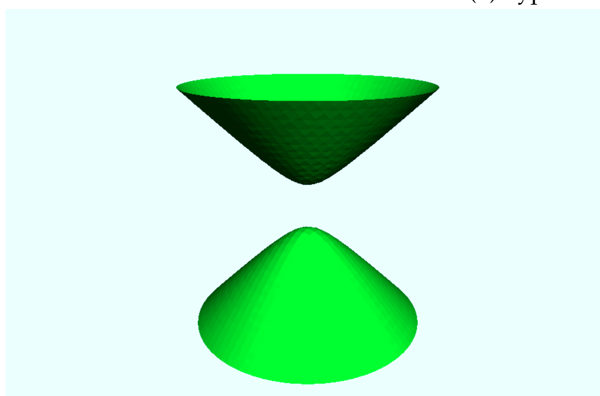
$$\frac{|y|}{|b|} = \frac{|z|}{|c|}.$$



(a) cone



(b) hyperboloid of one sheet



(c) hyperboloid of two sheets

In the case $k > 0$, the surface is a **hyperboloid of one sheet**. The surface intersects the xy-plane at the ellipse.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = k$$

In the case $k < 0$, the surface is a **hyperboloid of two sheets**. The surface does not intersect the xy-plane; it intersects the z-axis in two places where

$$\frac{z^2}{c^2} = -k.$$