Quadric Surfaces

In 3-dimensional space, we may consider quadratic equations in three variables $x$, $y$, and $z$:

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz + gx + hy + iz + j = 0$$

Such an equation defines a surface in 3D. Quadric surfaces are the surfaces whose equations can be, through a series of rotations and translations, put into quadratic polynomial equations of the form

$$\pm \frac{x^\alpha}{a^2} \pm \frac{y^\beta}{b^2} \pm \frac{z^\gamma}{c^2} = k$$

(1)

which are quadratic in at least two variables. That is, $\alpha$, $\beta$ and $\gamma$ are all either 1 or 2, and at least two are equal to 2.

There are six distinct types of quadric surfaces, arising from different forms of equation (1).

1. Ellipsoids

   The \textbf{ellipsoid} is the surface given by equations of the form

   $$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = k$$

   for positive $k$.

   The cross-sections are all ellipses.

   ![Figure 1: ellipsoid](image)

2. Elliptic paraboloids

   The \textbf{elliptic paraboloid} is the surface given by equations of the form

   $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z}{c} = 0.$$

   Cross-sections parallel to the $xy$-plane are ellipses, while those parallel to the $xz$- and $yz$-planes are parabolas.

   Note that the origin satisfies this equation.

3. Hyperbolic paraboloid
The **hyperbolic paraboloid** is the surface given by equations of the form

\[ \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z}{c} = 0. \]

Cross-sections parallel to the \( xy \)-plane are hyperbolas, while those parallel to the \( xz \)- and \( yz \)-planes are parabolas.

This curve has a shape similar to a saddle.

4. Cones, hyperboloids of one sheet and hyperboloids of two sheets

These surfaces all result from equations of the form

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = k \]

where \( k \) is a real constant. We can observe that for all such surfaces, cross-sections parallel to the \( xy \)-plane are ellipses, while cross-sections parallel to the \( yz \)-plane or the \( xz \)-plane are hyperbolas (or degenerate hyperbolas: a pair of intersecting lines).

In the case \( k = 0 \), the surface is a **cone**. We can observe that the surface contains the origin, and the intersection of the surface with, for instance, the \( yz \)-plane is the degenerate hyperbola

\[ \frac{|y|}{|b|} = \frac{|z|}{|c|}. \]
In the case \( k > 0 \), the surface is a **hyperboloid of one sheet**. The surface intersects the \( xy \)-plane at the ellipse.

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = k
\]

In the case \( k < 0 \), the surface is a **hyperboloid of two sheets**. The surface does not intersect the \( xy \)-plane; it intersects the \( z \)-axis in two places where

\[
\frac{z^2}{c^2} = -k.
\]