More examples with lines and planes

If two planes are not parallel, they will intersect, and their intersection will be a line. Given the equations of two non-parallel planes, we should be able to determine that line of intersection. Here is an example of doing just that.

Example: Suppose we want to find the intersection of the planes

$$P_1 : 3x - y + 6z = 5$$

and

$$P_2 : 2x + 3y - z = 10$$

Note that their normal vectors, $< 3, -1, 6 >$ and $< 2, 3, -1 >$ are not parallel (how can you tell?) so the planes are not parallel.

The key idea to finding the line of intersection is this: since the line lies in both planes, its direction vector is orthogonal to both planes' normal vectors.

Let L be the line of intersection. Then its direction vector is orthogonal to $< 3, -1, 6 >$ and $< 2, 3, -1 >$, and hence the direction vector is parallel to the cross product

$$< 3, -1, 6 > \times < 2, 3, -1 >=< -17, 15, 11 > \text{ (check!)}$$

So, we can take $< -17, 15, 11 >$ as the direction vector of L, and now we just need a point on the line.

Unless a line is parallel to the yz-plane, it will hit the yz-plane. Our line L is not parallel to the yz-plane (how do we know?), L hits the yz-plane; hence, there is a point on L at which $x = 0$, and, from the plane equations,

$$-y + 6z = 5$$

and

$$3y - z = 10$$

Those equations together give

$$y = \frac{65}{17} \quad \text{and} \quad z = \frac{25}{17}. \text{ (check!)}$$

Hence, the point $(0, 65/17, 25/17)$ is on L, so L has equations

$$x = -17t, y = \frac{65}{17} + 15t, z = \frac{25}{17} + 11t$$

Now, let's find the equation of the plane containing a given line, and a given point.

Example: Suppose we want the equation of the plane containing the line L from the last example, and the point $(5, 4, 3)$. If we had two vectors in the plane, then we could find the cross product, and use that as the normal to the plane. We already know $< -17, 15, 11 >$ is in the plane, and we know two points in the plane, so

$$\left\langle 5-0, 4-\frac{65}{17}, 3-\frac{25}{17}\right\rangle = \left\langle 5, \frac{3}{17}, \frac{26}{17}\right\rangle$$

is also in the plane.

The cross product gives us

$$< -17, 15, 11 > \times \left\langle 5, \frac{3}{17}, \frac{26}{17}\right\rangle =< 21, 81, -78 > \text{(check!)}$$

so the plane has equation

$$21(x-5) + 81(y-4) - 78(z-3) = 0$$

which can be rewritten as

$$21x + 81y - 78z - 195 = 0$$

or

$$21x + 81y - 78z = 195.$$

Another question that comes up regarding planes in 3D is that of finding the distance from a point to a plane. One way to solve such a problem is to find the line, orthogonal to the given plane, which passes through the given point. Then find the intersection of this line and the plane: the point of intersection will be the point on the plane closest to the given point, and the distance between these points is the distance from the point to the plane.

Example: Find the distance from the point $P = (3, -4, 7)$ to the plane $2x + 5y - z = 11$.

We begin by finding the parametric equations of the line L which passes through P and is orthogonal to the given plane. The direction vector for L will be parallel to the normal vector $< 2, 5, -1 >$ of the plane, and, in fact, we can take it to be just that. Hence, the line has parametric equations

$$x = 3 + 2t,\ y = -4 + 5t,\ z = 7 - t$$

The line will intersect the plane at the point specified by t if

$$2(3 + 2t) + 5(-4 + 5t) - (7 - t) = 11$$

which simplifies to

$$30t - 21 = 11$$

so $t = \dfrac{16}{15}$ gives the one point of intersection, which is

$$x = 3 + 2\frac{16}{15},\ y = -4 + 5\frac{16}{15},\ z = 7 - \frac{16}{15}$$

i.e., the point

$$Q = \left(\frac{77}{15}, \frac{4}{3}, \frac{89}{15}\right).$$

This is the point on the plane $2x + 5y - z = 11$ which is closest to the point $(3, -4, 7)$. The distance from Q to P is, via the distance formula,

$$\sqrt{\frac{512}{15}} = 5.84237394672....$$

Example: Let P be the plane $3x + 4y - z = 7$. Find two planes, parallel to P, that are each a distance of 3 units away from P.

Since P has normal vector $\langle 3, 4, -1\rangle$, the two parallel planes we are seeking have this as their normal vector as well. Hence, we only need to find a single point on each plane to nail down its equation.

To do that, we want to start with any point on P, and move a distance of 3 units in the direction, or the opposite direction, of $\langle 3, 4, -1\rangle$.

To do that, let's rescale the normal vector to get a vector of length 3. Such a vector $\vec{v}$ is

$$\vec{v} = 3 \cdot \frac{\langle 3, 4, -1\rangle}{\sqrt{3^2 + 4^2 + 1^2}} = \left\langle \frac{9}{\sqrt{26}}, \frac{12}{\sqrt{26}}, -\frac{3}{\sqrt{26}} \right\rangle.$$

Now, find a point on P. Any point will do. One can always (or almost always) simply pick x and y to be whatever you like, and then solve for z. Let's let $x = y = 0$, and then $z = -7$. So the point $(0, 0, -7)$ is in P.

Then, moving from $(0, 0, 7)$ along the vector $\vec{v}$ takes us to the point

$$\left(0 + \frac{9}{\sqrt{26}}, 0 + \frac{12}{\sqrt{26}}, 7 - \frac{3}{\sqrt{26}}\right) = \left(\frac{9}{\sqrt{26}}, \frac{12}{\sqrt{26}}, 7 - \frac{3}{\sqrt{26}}\right)$$

Hence, one of the planes we seek has equation

$$3\left(x - \frac{9}{\sqrt{26}}\right) + 4\left(y - \frac{12}{\sqrt{26}}\right) - \left(z - 7 + \frac{3}{\sqrt{26}}\right) = 0,$$

that is,

$$3x + 4y - z = 7 + \frac{78}{\sqrt{26}}.$$

Starting from $(0, 0, 7)$ and moving along the vector $-\vec{v}$ takes us to the point

$$\left(-\frac{9}{\sqrt{26}}, -\frac{12}{\sqrt{26}}, 7 + \frac{3}{\sqrt{26}}\right)$$

and this yields the other paralle plane,

$$3\left(x+\frac{9}{\sqrt{26}}\right)+4\left(y+\frac{12}{\sqrt{26}}\right)-\left(z-7-\frac{3}{\sqrt{26}}\right)=0,$$

that is,

$$3x+4y-z=7-\frac{78}{\sqrt{26}}.$$