Example of a volume question involving a quadric surface

Suppose we are interested in finding the volume bounded by the hyperboloid $-x^2 - y^2 + z^2 = 25$ and the plane $z = 10$.

By writing the hyperboloid equation as

$$z^2 = 25 + x^2 + y^2$$

we can more easily see that it is a hyperboloid of two sheets, one sheet above the xy-plane and one below it. Above the xy-plane, we see the z value starts at a minimum of 5 above the origin and increases as the point (x, y) moves away from the origin.

By setting $z = 10$ in the above equation, we can find the intersection of the surface and the plane $z = 10$:

$$\begin{aligned} 10^2 &= 25 + x^2 + y^2 \\ 75 &= x^2 + y^2 \end{aligned}$$

So we see that the intersection is a circle that lies above (at $z = 10$) the circle in the xy-plane with radius $\sqrt{75}$ centered at the origin.

As a result, we can discern that the 3D region of space we are find the volume of is located below the plane $z = 10$ and above the surface $z^2 = 25 + x^2 + y^2$.

All of the points in this 3D region lie above the disk

$$x^2 + y^2 \le 75.$$

The volume we seek is therefore the integral of $10 - \sqrt{25 + x^2 + y^2}$ over this disk.

Since the disk is easily described using polar coordinates, and the integerand involves $x^2 + y^2 = r^2$, we try using polar coordinates to evaluate the integral.

We have

$$\begin{aligned} \text{volume } &= \int_0^{2\pi} \int_0^{\sqrt{75}} (10 - \sqrt{25 + r^2})\, r \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^{\sqrt{75}} (10r - r(25 + r^2)^{1/2}) \, dr \, d\theta \\ &= \int_0^{2\pi} 5r^2 - \frac{1}{3}(25 + r^2)^{3/2} \Big|_0^{\sqrt{75}} \, d\theta \\ &= \int_0^{2\pi} \frac{250}{3} \, d\theta \\ &= \frac{500}{3}\pi. \end{aligned}$$