Math 126G - Spring 2002
First Mid-Term Exam Solutions
April 23, 2002

1. Indicate whether each of the following statements is true or false by circling T or F.

T If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

F If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ must converge.

T If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=100}^{\infty} a_n$ must converge.

F If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ diverge, then $\sum_{n=1}^{\infty}(a_n + b_n)$ must diverge.

F If $\lim_{n\to\infty} a_n = L$, then $\sum_{n=1}^{\infty} a_n = L$.

2. Determine whether each sequence converges or diverges. If it converges, find the limit.

(a) $\left\{(-1)^n \sin\left(\frac{2}{n}\right)\right\}$

$$\lim_{n\to\infty} |(-1)^n \sin\left(\frac{2}{n}\right)| = \lim_{n\to\infty} |\sin\left(\frac{2}{n}\right)| = |\sin(\lim_{n\to\infty} \frac{2}{n})| = |\sin(0)| = 0,$$

so

$$\lim_{n\to\infty} (-1)^n \sin\left(\frac{2}{n}\right) = 0.$$

So, the sequence converges.

(b) $\left\{\left(n + \frac{1}{n}\right)^2 - n^2\right\}$

$$\left(n + \frac{1}{n}\right)^2 - n^2 = \lim_{n\to\infty} \left(n^2 + 2 + \frac{1}{n^2} - n^2\right) = \lim_{n\to\infty} \left(2 + \frac{1}{n^2}\right)$$

$= 2 + 0 = 2$, so the sequence converges.

3. Determine whether the series $\sum_{n=2}^{\infty}(-1)^n \frac{3}{4^n}$ converges or diverges. If it converges, find its sum.

This is a geometric series with first term 3/16 and common ratio −1/4. Since $|-1/4| = 1/4 < 1$, the series converges, and the sum is

$$\frac{\frac{3}{16}}{1-(-\frac{1}{4})} = \frac{\frac{3}{16}}{\frac{5}{4}} = 3/20.$$

4. By comparing it with an integral, give an upper bound for the series $\sum_{n=3}^{\infty} \frac{1}{n(\ln n)^2}$. That is, find a value A so that

$$\sum_{n=3}^{\infty} \frac{1}{n(\ln n)^2} < A.$$

Since n strictly increases with n, and $\ln n$ strictly increases with n, $n(\ln n)^2$ strictly increases with n. Hence, $\frac{1}{n(\ln n)^2}$ strictly decreases with n. Also, $\frac{1}{n(\ln n)^2} > 0$ for $n > 1$. As a result, we have the comparison

$$\sum_{n=3}^{\infty} \frac{1}{n(\ln n)^2} < \int_2^{\infty} \frac{dx}{x(\ln x)^2}$$

provided the limit converges. We find

$$\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \lim_{t\to\infty} \int_2^t \frac{dx}{x(\ln x)^2} = \lim_{t\to\infty} \int_{\ln 2}^t \frac{du}{u^2}$$

$$= \lim_{t\to\infty} -\frac{1}{u}\bigg|_{\ln 2}^t = \lim_{t\to\infty}\left(-\frac{1}{t} + \frac{1}{\ln 2}\right) = 0 + \frac{1}{\ln 2} = \frac{1}{\ln 2}.$$

So $A = \frac{1}{\ln 2}$ is an upper bound for the series.

5. Determine whether each of the following series converge or diverge. Explain your answer and show all work.

   (a) $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

   This series diverges. You should show this using the limit comparison test with the harmonic series. The integral test would be another more complicated option.

(b) $\sum_{n=0}^{\infty} \frac{2^n}{(n!)^2}$

This series converges. You should show this using the ratio test:

$$\lim_{n\to\infty} \frac{2^{n+1}}{((n+1)!)^2} \frac{(n!)^2}{2^n} = \lim_{n\to\infty} \frac{2}{(n+1)^2} = 0 < 1.$$

6. Consider the power series

$$\sum_{n=0}^{\infty} (-2)^n \frac{x^n}{n+1}.$$

(a) Find all values of x for which the series converges.

Using the ratio test, we have

$$\lim_{n\to\infty} \left| \frac{(-2)^{n+1} x^{n+1}}{n+2} \frac{n+1}{(-2)^n x^n} \right| = \lim_{n\to\infty} \left| \frac{2x(n+1)}{n+2} \right| = \lim_{n\to\infty} |2x|,$$

so the series converges absolutely if $|2x| < 1$ and diverges if $|2x| > 1$. If $|2x| = 1$, then $x = 1/2$ or $x = -1/2$. For $x = 1/2$, we have

$$\sum_{n=0}^{\infty} (-2)^n \frac{x^n}{n+1} = \sum_{n=0}^{\infty} (-2)^n \frac{(1/2)^n}{n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1}$$

which you can show is convergent by the alternating series test. If $x = -1/2$, we have

$$\sum_{n=0}^{\infty} (-2)^n \frac{x^n}{n+1} = \sum_{n=0}^{\infty} (-2)^n \frac{(-1/2)^n}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1}$$

which is the harmonic series and is therefore divergent. Thus this power series converges for $\frac{-1}{2} < x \leq \frac{1}{2}$.

(b) What is the radius of convergence of this series?

Since the interval of convergence has length 1, the radius of convergence is 1/2.

7. Determine a power series for the function $f(x) = \frac{x^2}{1-2x}$.

We know

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

so

$$\frac{1}{1-2x} = \sum_{n=0}^{\infty} (2x)^n$$

so

$$\frac{x^2}{1-2x} = x^2 \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} x^2 (2x)^n = \sum_{n=0}^{\infty} 2^n x^{n+2} = \sum_{n=2}^{\infty} 2^{n-2} x^n = \sum_{n=2}^{\infty} \frac{2^n}{4} x^n$$