Math 126 C - Winter 2006
Mid-Term Exam Number Two
Solutions
February 16, 2006

1. *Eliminate the parameter in the following parametric equation pair to get a Cartesian equation for the curve that involves no trigonometric functions.*

$$x = \cos t, y = \sin t - \cos t$$

There are many different ways to solve this. Here's one:

We know

$$\sin^2 t + \cos^2 t = 1$$

and

$$\cos t = x$$

and

$$\sin t = y + \cos t = y + x$$

so that

$$(y + x)^2 + x^2 = 1$$

and we're done.

Here's another way: notice that we can write

$$\sin t = \pm\sqrt{1 - \cos^2 t} = \pm\sqrt{1 - x^2}$$

so that

$$y = \pm\sqrt{1 - x^2} - x.$$

2. *Consider the curve defined parametrically by the parametric equations*

$$x = \ln \ln t, y = \ln t - (\ln t)^2.$$

Find the equation of the tangent line to the curve at the point $t = e$.

If $t = e$ then $x = 0$, and $y = 0$. We have

$$\frac{dx}{dt} = \frac{1}{t \ln t} = \frac{1}{e}$$

when $t = e$, and

$$\frac{dy}{dt} = \frac{1}{t} - 2(\ln t)\frac{1}{t} = \frac{1}{e} - \frac{2}{e} = -\frac{1}{e}$$

when $t = e$.

Thus, when $t = e$,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -1$$

so the tangent line has equation $y = -x$.

3. *Find the parametric equations for the tangent line to the curve defined by*

$$x = t^3 - t, y = t^6 + t^2 + 1, z = \frac{1}{2}t^2 + 5t$$

at the point $(0, 1, 0)$.

We find

$$\frac{dx}{dt} = 3t^2 - 1$$

$$\frac{dy}{dt} = 6t^5 + 2t$$

$$\frac{dz}{dt} = t + 5$$

The point $(0, 1, 0)$ corresponds to $t = 0$. To see this, note that we need

$$t^3 - t = 0$$

which tells us

$$t(t^2 - 1) = 0$$

so we know $t = 0$, $t = 1$ or $t = -1$. Checking these values with $y = t^6 + t^2 + 1$, we find that only $t = 0$ works.

Plugging $t = 0$ into the derivatives above, and using the point $(0, 1, 0)$, we have the tangent line equations

$$x = -t, y = 1, z = 5t.$$

4. *At what point does the curve* $y = e^x$ *have maximum curvature?*

We have a formula to find the curvature function $\kappa(x)$ for the graph of a given function $f(x) = e^x$:

$$\kappa(x) = \frac{|f''(x)|}{(1 + (f'(x))^2)^{3/2}} = \frac{e^x}{(1 + e^{2x})^{3/2}}$$

Note that we have used the fact that $e^x > 0$ for all x to remove the absolute value symbol.

Now we want to find out how large this function $\kappa(x)$ can get. We can start our search for maxima of this function be studying the function's first derivative.

$$\kappa'(x) = \frac{e^x(1 + e^{2x})^{3/2} - e^x\frac{3}{2}(1 + e^{2x})^{1/2}2x}{(1 + e^{2x})^3}$$

$$= \frac{e^x(1 + e^{2x})^{1/2}\left((1 + e^{2x}) - 3e^{2x}\right)}{(1 + e^{2x})^3} = \frac{e^x(1 + e^{2x})^{1/2}(1 - 2e^{2x})}{(1 + e^{2x})^3}$$

We note that this is defined for all x, so the only critical points will occur where this is zero. If this is zero, then

$$1 - 2e^{2x} = 0$$

since $e^x > 0$ for all x, and $(1+e^{2x})^{1/2} > 0$ for all x.

Hence, the only critical point is at

$$x = \frac{1}{2}\ln\frac{1}{2}.$$

Since $2e^{2x}$ is a strictly increasing function, we can see that that $\kappa'(x)$ is going to be negative for x greater than the value we just found, and it will be positive for x less than that value. In other words, we can conclude that this value of x gives us the maximum value of $\kappa(x)$.

The point on the curve where curvature is maximum is thus

$$\left(\frac{1}{2}\ln\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$$

5. Find the length of the curve defined by

$$\vec{r}(t) = \left\langle \frac{2\sqrt{2}}{3}t^{3/2}, t, \frac{1}{2}t^2 \right\rangle, 0 \le t \le 4$$

Conveniently,

$$\left(\frac{d}{dt}\frac{2\sqrt{2}}{3}t^{3/2}\right)^2 + \left(\frac{d}{dt}t\right)^2 + \left(\frac{d}{dt}\frac{1}{2}t^2\right)^2 = (t+1)^2$$

so the arc length is just

$$\int_0^4 (t+1)\,dt = 12.$$

6. *Find the curvature of the curve defined by*

$$\vec{r}(t) = \left\langle \frac{1}{2}t^2 - 2t, t^2 - t, t^2 + t \right\rangle$$

at the point $t = 0$.

We have two useful formulas for finding the curvature of a 3D curve. One is

$$\kappa(t) = \frac{|\vec{T}'(t)|}{|r'(t)|}$$

and the other is

$$\kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)}{|\vec{r}'(t)|^3}$$

For this problem, the second equation is **much** easier to use. I really can't think of a time when it would be preferable to use the first one, expect in that rare occasion where you are given $\vec{T}'(t)$ and don't have to derive it from $\vec{r}(t)$.

So we use the second formula. Since we are interested in $\kappa(0)$, we need only find $\vec{r}\,'(0)$ and $\vec{r}\,''(0)$ and plug them into the formula:

$$\vec{r}\,'(t) = \langle t-2, 2t-1, 2t+1\rangle$$

$$\vec{r}\,'(0) = \langle -2, -1, 1\rangle$$
$$\vec{r}\,''(t) = \langle 1, 2, 2\rangle$$
$$\vec{r}\,''(0) = \langle 1, 2, 2\rangle$$

Plugging these into our formula gives us

$$\kappa(0) = \frac{\sqrt{50}}{(\sqrt{6})^3} = \frac{5}{6\sqrt{3}}.$$