Math 126 C - Winter 2006
Mid-Term Exam Number One
January 31, 2006
Solutions

1. It is necessary to find the first two derivatives of $f(x)$:

$$f'(x) = \frac{1}{x \ln x}$$

$$f''(x) = -\frac{1 + \ln x}{(x \ln x)^2}$$

   Evaluating f, f', and f'' at $x = e$ gives the coefficients of $T_2(x)$ for $f(x)$:

$$T_2(x) = \frac{1}{e}(x - e) - \frac{1}{e^2}(x - e)^2$$

2. The first four non-zero terms of the Taylor series for $\sin x$ are

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

   so the first four non-zero terms of the Taylor series for $\sin x^2$ are

$$x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!}$$

   Integrating this from 0 to 2 gives

$$\left(\frac{1}{3}x^3 - \frac{1}{7 \cdot 3!}x^7 + \frac{1}{11 \cdot 5!}x^{11} - \frac{1}{15 \cdot 7!}x^{15}\right)\Big|_0^2 = \frac{8}{3} - \frac{128}{42} + \frac{2048}{1320} - \frac{32768}{75600} = 0.7371236.$$

3. (Version 1) We use

$$\frac{1}{1 - x} = 1 + x + x^2 + x^3 + ...$$

   so that

$$\frac{1}{1 - (-5x)} = 1 - 5x + (5x)^2 - (5x)^3 + ...$$

   and

$$\frac{1}{3 + x} = \left(\frac{1}{3}\right)\frac{1}{1 - (-\frac{1}{3}x)} = \frac{1}{3}\left(1 - \frac{1}{3}x + \left(\frac{1}{3}x\right)^2 - \left(\frac{1}{3}x\right)^3 + ...\right) = \frac{1}{3} - \frac{1}{9}x + \frac{1}{27}x^2 - \frac{1}{81}x^3 + ...$$

   Adding these together, we have

$$\frac{1}{1 + 5x} + \frac{1}{3 + x} = \frac{4}{3} - \frac{46}{9}x + \frac{676}{27}x^2 - \frac{10126}{81}x^3 + ...$$

(Version 2) Using a similar technique,

$$\frac{1}{1+7x} = 1 - 7x + (7x)^2 - (7x)^3 + ...$$

and

$$\frac{1}{2+x} = \left(\frac{1}{2}\right)\frac{1}{1+\frac{x}{2}} = \frac{1}{2}\left(1 + \frac{x}{2} + \left(\frac{x}{2}\right)^2 + \left(\frac{x}{2}\right)^3 + ...\right)$$

Adding these together, we have

$$\frac{1}{2+x} + \frac{1}{1+7x} = \frac{3}{2} - \frac{27}{4}x + \frac{393}{8}x^2 - \frac{5489}{16}x^3 + ...$$

4. (Version 1)

$$\theta = \cos^{-1}\left(\frac{4}{\sqrt{26}\sqrt{11}}\right) = 1.80958408790828020.$$

(Version 2)

$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{89}\sqrt{69}}\right) = 1.5452718055317992.$$

5. Since the vectors are orthogonal,

$$< x, 3, 2 > \cdot < 2, 3, x >= 0$$

so

$$2x + 9 + 2x = 0$$

from which we conclude that $x = -\frac{9}{4}$.

6. Finding vectors parallel (or "in") the plane, we get

$$\vec{a} =< 4, 3, -3 > \text{ and } \vec{b} =< 1, -12, 6 >$$

Their dot product is a normal vector to the plane:

$$\vec{a} \times \vec{b} =< -18, -27, -51 >$$

so the equation of the plane is

$$-18(x + 3) - 27(y - 4) - 51z = 0.$$

7. (Version 1) The direction vector of the line is orthogonal to the normal vectors of both planes so is parallel to

$$< 1, 1, -1 > \times < 2, -3, 4 >=< 1, -6, -5 >$$

This vector is parallel to the plane we're looking for. To get another vector, we need another point in the plane. We can get a point that's on the line, and hence on the plane, by setting $x = 0$ and solving. We find $(0, 17, 14)$ is on the line, and on the plane. The vector extending from this point to $(-2, 7, 3)$ is

$$< -2, -10, -11 >$$

and so the normal vector to the plane we seek is parallel to

$$< -2, -10, -11 > \times < 1, -6, -5 >=< -16, -21, 22 >$$

So, the equation of the plane is

$$-16(x + 2) - 21(y - 7) + 22(z - 3) = 0$$

(Version 2) The direction vector of the line is orthogonal to the normal vectors of both planes so is parallel to

$$< 1, 1, -1 > \times < 2, -3, 4 >=< 1, -6, -5 >$$

This vector is parallel to the plane we're looking for. To get another vector, we need another point in the plane. We can get a point that's on the line, and hence on the plane, by setting $x = 0$ and solving. We find $(0, 22, 16)$ is on the line, and on the plane. The vector extending from this point to $(-2, 7, 3)$ is

$$< -2, -15, -13 >$$

and so the normal vector to the plane we seek is parallel to

$$< 1, -6, -5 > \times < -2, -15, -13 >=< 3, 23, -27 >$$

So, the equation of the plane is

$$3x + 23(y - 22) - 27(x - 16) = 0.$$