

## Euler's Formula via Taylor Series Worksheet

In this worksheet, you will prove the formula

$$e^{i\theta} = \cos \theta + i \sin \theta.$$

This is perhaps the most famous of all formulas in mathematics. It is known as **Euler's formula**, for Leonard Euler, a Swiss mathematician who lived from 1707 to 1783. The formula was actually first proved by Roger Cotes in 1714, an English mathematician who lived from 1682 to 1716.

The  $i$  in the formula is known as the **imaginary unit**. It is defined to be the non-real number with the property that

$$i^2 = -1.$$

Since  $i$  is not a real number, it is said to be **imaginary**, and it gives rise to the set of **complex numbers**. Complex numbers are numbers of the form

$$a + bi$$

where  $a$  and  $b$  are real numbers. Euler's formula expresses an equality between two ways of representing a complex number.

You can use Taylor series to prove the formula.

Here are a few steps.

1. The first thing to do is to check out what happens to powers of  $i$ . Since

$$i^2 = -1,$$

we have  $i^3 = -i$ . What is  $i^4$ ?  $i^5$ ?  $i^{62}$ ? What is  $i^n$  for a general positive integer  $n$ ?

2. The Taylor series for  $e^x$ ,

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

converges for all real  $x$ . In fact, for any *complex* number  $x$ , the series converges to  $e^x$ .

Use what you learned in step 1 to write out the first 8 (or more) terms of the series for

$$e^{i\theta}.$$

3. How does this compare to the Taylor series for  $\cos x$  and  $\sin x$ ? Show how this gets us Euler's formula.

This is a bit of a casual proof. By getting a general expression for the  $n$ -th term of the series for  $e^{i\theta}$ , and our knowledge of the  $n$ -th term of the series for  $\cos \theta$  and  $\sin \theta$ , the proof could be made completely solid.

### What can you do with Euler's formula?

1. If you let  $\theta = \pi$ , Euler's formula simplifies to what many claim is the most beautiful equation in all of mathematics:

$$e^{i\pi} = -1.$$

It does tie together three important constants,  $e$ ,  $i$ , and  $\pi$  rather nicely.

2. We can get quick proofs for some trig identities from Euler's formula. We need this fact: if  $a$ ,  $b$ ,  $c$ , and  $d$  are real numbers, and

$$a + bi = c + di$$

then  $a = c$  and  $b = d$ . That is, if two complex numbers are equal, then their real parts are equal and their imaginary parts are equal.

Now, replacing  $\theta$  by  $n\theta$  in Euler's formula we have

$$e^{in\theta} = \cos(n\theta) + i \sin(n\theta)$$

However, the left side can be written as

$$e^{in\theta} = (e^{i\theta})^n = (\cos \theta + i \sin \theta)^n$$

3. Let  $n = 2$  and expand to prove the two double-angle formulas

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

and

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

4. Let  $n = 3$  and expand to prove the less common triple-angle formulas

$$\cos 3\theta = \cos \theta (\cos^2 \theta - 3 \sin^2 \theta) = \cos \theta (4 \cos^2 \theta - 3)$$

and

$$\sin 3\theta = \sin \theta (3 \cos^2 \theta - \sin^2 \theta) = \sin \theta (3 - 4 \sin^2 \theta)$$

You can see that, by letting  $n$  be other integers, many other formulas are possible.