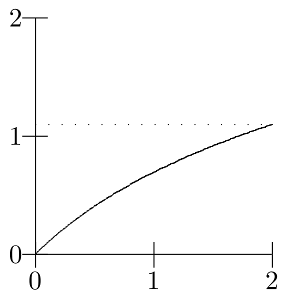
Math 125G - Spring 2002
Second Mid-Term Exam Solutions
May 21, 2002

1. (20 points) Consider the region bounded by $y = \ln(x+1)$, the $y$-axis, and the line $y = \ln 3$. Suppose we revolve this region about the $y$-axis to create a solid in three-dimensional space.




   (a) Set up an integral representing the volume of this solid. **Do not evaluate the integral.**

   Volume $= \int_0^2 2\pi x(\ln 3 - \ln(x+1))\ dx.$

   (b) Suppose this solid is a tank filled with a liquid that weighs 45 lb/ft$^3$. If the linear units are feet, set up an integral representing the work done in moving all of the liquid to the top of the tank. **Do not evaluate the integral.**

   Work $= \int_0^{\ln 3} 45\pi(e^y - 1)^2(\ln 3 - y)\ dy.$

2. (10 points) Let $0 < p < 1$. Find the average value of the function $x^p(1 - x^{1/p})$ on the interval $[0, 1]$.

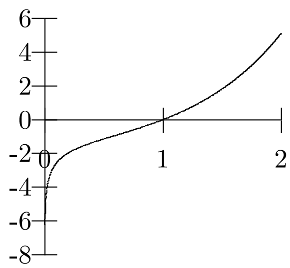
The average value is

$$\frac{1}{1-0}\int_0^1 x^p(1 - x(1/p))\ dx = \int_0^1 (x^p - x^{p+1/p})\ dx =$$

$$\left(\frac{x^{p+1}}{p+1} - \frac{x^{p+1/p}}{p+1/p+1}\right)\Bigg|_0^1 = \frac{1}{p+1} - \frac{1}{p+1/p+1} = \frac{1}{p+1} - \frac{p}{p^2+p+1}$$

$$\frac{p^2+p+1-p(p+1)}{(p+1)(p^2+p+1)} = \frac{1}{(p+1)(p^2+p+1)}.$$

3. (10 points) Use Simpson's rule with $n = 4$ to estimate the volume of the solid created by revolving the region bounded by $y = e^x \ln x$, the $x$-axis, and $x = 2$ about the $x$-axis. Your answer should be correct to at least 6 digits, so use all the digits you get.




The integral that we are interested in is

$$I = \int_1^2 \pi (e^x \ln x)^2 \; dx.$$

Let $f(x) = \pi(e^x \ln x)^2$. Using Simpson's rule, we have

$$I \approx \frac{1}{3 \cdot 4} \left( f(1.0) + 4f(1.25) + 2f(1.5) + 4f(1.75) + f(2.0) \right)$$

$$= \frac{1}{12}(0 + 4(1.9057009) + 2(10.37385786) + 4(32.580704885) + 82.40977385)$$

$$= 20.09192608884$$

4. (20 points) Evaluate the following integrals.

(a) $$\int \frac{dx}{x^2 + 4x + 13}$$

$$\int \frac{dx}{x^2 + 4x + 13} = \int \frac{dx}{(x+2)^2 + 9}.$$

Let $x + 2 = 3 \tan u$, so $dx = 3 \sec^2 u \; du$, and the integral becomes

$$\int \frac{3 \sec^2 u}{9 \tan^2 u + 9} \; du = \int \frac{3 \sec^2 u}{9 \sec^2 u} \; du$$

$$= \int \frac{1}{3} \; du = \frac{1}{3} u + C = \frac{1}{3} \tan^{-1} \left( \frac{x+2}{3} \right) + C.$$

(b) $$\int \frac{dx}{x^2 + 7x + 12}$$

$$\int \frac{dx}{x^2 + 7x + 12} = \int \frac{dx}{(x+3)(x+4)}$$

We set

$$\frac{1}{(x+3)(x+4)} = \frac{A}{x+3} + \frac{B}{x+4}$$

and solve for $A$ and $B$. We find $A = 1$, $B = -1$, so our integral becomes

$$\int \left( \frac{1}{x+3} - \frac{1}{x+4} \right) dx = \ln|x+3| - \ln|x+4| + C.$$

5. (20 points) Evaluate the following integrals.

(a) $$\int \frac{x^5}{\sqrt{1+x^2}}\ dx$$

Letting $x = \tan u$, so that $dx = \sec^2 u\ du$, the integral becomes

$$\int \frac{\tan^5 u \sec^2 u}{\sqrt{1+\tan^2 u}}\ du = \int \frac{\tan^5 u \sec^2 u}{\sec u}\ du = \int \tan^5 u \sec u\ du =$$

$$\int \tan^4 u \sec u \tan u\ du = \int \left(\sec^2 u - 1\right)^2 \sec u \tan u\ du$$

Now let $w = \sec u$, so $dw = \sec u \tan u\ du$, and the integral becomes

$$\int \left(w^2-1\right)^2\ dw = \int (w^4 - 2w^2 + 1)\ dw = \frac{w^5}{5} - \frac{2w^3}{3} + w + C$$

Since $w = \sec u$, and $x = \tan u$, $w = \sqrt{x^2+1}$. Hence, our integral equals

$$\frac{1}{5}(x^2+1)^{\frac{5}{2}} - \frac{2}{3}(x^2+1)^{\frac{3}{2}} + \sqrt{x^2+1} + C = \frac{1}{15}(3x^4 - 4x^2 + 8)\sqrt{x^2+1} + C.$$

(b) $$\int \sin\sqrt{x}\ dx$$

Letting $w = \sqrt{x}$, we have $dw = \frac{1}{2}x^{-\frac{1}{2}}\ dx$, i.e., $2wdw = dx$. So, the integral becomes

$$\int 2w \sin w\ dw = 2\int w \sin w\ dw.$$

We apply integration by parts with $u = w, dv = \sin w$, so $du = dw, v = -\cos w$ and we have

$$2\int w \sin w\ dw = 2\left(-w\cos w + \int \cos w\ dw\right) = 2\left(-w\cos w + \sin w\right) + C$$

$$= -2w\cos w + 2\sin w + C = -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x}$$

6. (10 points) Evaluate the following integral:

$$\int_0^\infty x^2 e^{-x}\ dx$$

This is an inproper integral, but let's worry about that later. First, find an antiderivative. We use integration by parts:

$$\int x^2 e^{-x}\ dx = -x^2e^{-x} + 2\int xe^{-x}\ dx = -x^2e^{-x} + 2\left(-xe^{-x} + \int e^{-x}\ dx\right) =$$

$$-x^2e^{-x} + 2\left(-xe^{-x} - e^{-x}\right) + C = -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C.$$

Now, as $x$ goes to infinity, $e^{-x}$, $xe^{-x}$, and $x^2e^{-x}$ all go to zero, so

$$\int_0^\infty x^2e^{-x}\ dx = \lim_{t\to\infty}\left(-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right)\Big|_0^t = 2.$$