Math 125U - Winter 2002
Mid-Term Exam Solutions
February 6, 2002

1. Is $\dfrac{x}{x^2+1}$ an antiderivative of $\dfrac{1-x^2}{(x^2+1)^2}$? Explain.

$$\frac{d}{dx}\left(\frac{x}{x^2+1}\right) = \frac{(1)(x^2+1) - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$

so, yes, $\dfrac{x}{x^2+1}$ is an antiderivative of $\dfrac{1-x^2}{(x^2+1)^2}$.

2. Suppose $f''(x) = 4x+1$, $f'(1) = 2$, and $f(0) = 1$. Find f.

Integrating $f''(x) = 4x + 1$, we find $f'(x) = 2x^2 + x + C$.
Since $f'(1) = 2$, we have $2(1)^2 + (1) + C = 2$, i.e., $3 + C = 2$, so $C = -1$.
Hence $f'(x) = 2x^2 + x - 1$.
Integrating again, we have $f(x) = \frac{2}{3}x^3 + \frac{1}{2}x^2 - x + C_2$.
We know $f(0) = 1$, so $C_2 = 1$, and hence $f(x) = \frac{2}{3}x^3 + \frac{1}{2}x^2 - x + 1$.

3. The velocity of a rocket is measured every half-second after lift-off. The data is in the following table.

t	0.0	0.5	1.0	1.5	2.0	2.5
$v(t)$ (m/s)	0.0	1.3	2.7	5.3	12.0	22.3

Assuming the velocity was strictly increasing, find best possible lower and upper estimates for the height of the rocket (assuming an initial height of zero) after 2.5 seconds.

The under estimate is determined by assuming that the velocity of the rocket during each 0.5 second interval is given by the velocity at the start of the interval. So the distance travelled during the first 2.5 seconds is at least

$$0.0(0.5) + 1.3(0.5) + 2.7(0.5) + 5.3(0.5) + 12.0(0.5) = 10.65.$$

Similarly, an upper bound for the height is given by

$$1.3(0.5) + 2.7(0.5) + 5.3(0.5) + 12.0(0.5) + 22.3(0.5) = 21.8.$$

4. Suppose

$$\int_2^{10} f(x)\,dx = 12, \quad \int_2^{6} f(x)\,dx = -4, \quad \int_5^{10} f(x)\,dx = 1.$$

Find $\displaystyle\int_5^6 f(x)\,dx$.

There are a few ways to approach this; one way is the following.

$$\int_5^6 f(x)\,dx + \int_6^{10} f(x)\,dx = \int_5^{10} f(x)\,dx$$

so

$$\int_5^6 f(x)\,dx + \int_2^{10} f(x)\,dx - \int_2^6 f(x)\,dx = \int_5^{10} f(x)\,dx$$

i.e.,

$$\int_5^6 f(x)\,dx + 12 - (-4) = 1$$

so

$$\int_5^6 f(x)\,dx = 1 - 16 = -15.$$

5. Suppose $g(x) = \displaystyle\int_3^{\ln x} \frac{\ln t}{e^t}\,dt$. Find $g'(x)$.

   Let

$$u = \ln x$$

   and let

$$y = g(x) = \int_3^u \frac{\ln t}{e^t}\,dt.$$

   Then, by the Fundamental Theorem of Calculus, Part I,

$$\frac{dy}{du} = \frac{\ln u}{e^u}.$$

   By the chain rule, we have

$$g'(x) = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{\ln u}{e^u}\frac{1}{x} = \frac{\ln \ln x}{e^{\ln x}}\frac{1}{x} = \frac{\ln \ln x}{x^2}.$$

6. Evaluate the integrals.

   (a) $\displaystyle\int e^x \cos(2e^x)\,dx$

   Let $u = 2e^x$, so $du = 2e^x\,dx$, i.e., $\frac{1}{2}du = e^x\,dx$. Then

$$\int e^x \cos(2e^x)\,dx = \int \frac{1}{2}\cos u\,du = \frac{1}{2}\sin u + C = \frac{1}{2}\sin(2e^x) + C.$$

   (b) $\displaystyle\int (1+t)\sqrt{2+t}\,dt$

   Let $u = 2 + t$ so $du = dt$, and $1 + t = u - 1$. Then

$$\int (1+t)\sqrt{2+t}\,dt = \int (u-1)u^{\frac{1}{2}}\,du = \int (u^{\frac{3}{2}} - u^{\frac{1}{2}})\,du = \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + C =$$

$$\frac{2}{5}(2+t)^{\frac{5}{2}} - \frac{2}{3}(2+t)^{\frac{3}{2}} + C.$$

7. Evaluate the integrals.

(a) $\int x^9\sqrt{x^5-2}\,dx$

Let $u = x^5 - 2$. Then $du = 5x^4\,dx$, $u+2 = x^5$, so $u+2\,du = 5x^9\,dx$. Thus, $\frac{1}{5}(u+2)\,du = x^9\,dx$. So

$$\int x^9\sqrt{x^5-2}\,dx = \int \frac{1}{5}(u+2)u^{\frac{1}{2}}\,du = \frac{1}{5}\int (u^{\frac{3}{2}} + 2u^{\frac{1}{2}})\,du = \frac{1}{5}(\frac{2}{5}u^{\frac{5}{2}} + \frac{4}{3}u^{\frac{3}{2}}) + C =$$

$$\frac{2}{25}(x^5-2)^{\frac{5}{2}} + \frac{4}{15}(x^5-2)^{\frac{3}{2}} + C.$$

(b) $\int \frac{1}{x\sqrt{\ln x}}\,dx$

Let $u = \ln x$, so $du = \frac{1}{x}\,dx$. Then

$$\int \frac{1}{x\sqrt{\ln x}}\,dx = \int \frac{1}{\sqrt{u}}\,du = \int u^{-\frac{1}{2}}\,du = 2u^{\frac{1}{2}} + C = 2\sqrt{\ln x} + C.$$

8. Find the area of the region bounded by $y = 2x$ and $y = x^2 - 3x$.

First we find the intersection points of these two curves. Setting $2x = x^2 - 3x$, we have $0 = x^2 - 5x = x(x-5)$ so $x = 0$ and $x = 5$ give the two points of intersection. Since if $x = 1$, $2x > x^2 - 3x$, we can conclude that $2x \geq x^2 - 3x$ for all x in the interval $0 \leq x \leq 5$. Hence, the area of the region bounded by these two curves is

$$\int_0^5 (2x - (x^2 - 3x))\,dx = \int_0^5 (5x - x^2)\,dx = \frac{5}{2}x^2 - \frac{1}{3}x^3\Big|_0^5 = \frac{5}{2}(25) - \frac{1}{3}(125) = \frac{125}{6}.$$

9. Find the volume of the solid created by revolving the region bounded by

$$y = e^{x^2},\ x = \sqrt{\ln(\pi+1)},\ x = 0,\quad \text{and } y = 0$$

about the y-axis.

The region is bounded on the left by the y-axis, on the right by the vertical line $x = \sqrt{\ln(\pi+1)}$, below by the x-axis, and above by $y = e^{x^2}$. Hence, the volume obtained by revolving this region about the y-axis is

$$V = \int_0^{\sqrt{\ln(\pi+1)}} 2\pi x e^{x^2}\,dx.$$

Letting $u = x^2$, so that $du = 2x\,dx$, we have

$$V = \int_0^{\ln(\pi+1)} \pi e^u\,du = \pi e^u\big|_0^{\ln(\pi+1)} = \pi\left(e^{\ln(\pi+1)} - e^0\right) = \pi(\pi+1-1) = \pi^2.$$

10. Consider the region bounded by the curve $y = \ln x$ and the line which passes through $(1,0)$ and $(e,1)$.

   (a) Set up (but do not evaluate) an integral representing the volume of the solid obtained by revolving this region about the y-axis.

   The line has a slope of

$$\frac{1-0}{e-1} = \frac{1}{e-1}$$

   so the line has equation

$$y = \frac{1}{e-1}(x-1) = \frac{x-1}{e-1}.$$

   The line intersects the curve at $(1,0)$ and $(e,1)$. The line lies below the curve $y = \ln x$ (since $y = \ln x$ is always concave down, or by checking any value of x between 1 and e), so the volume is (using the shell method)

$$V = \int_1^e 2\pi x \left(\ln x - \frac{x-1}{e-1} \right) dx.$$

   (b) Set up (but do not evaluate) an integral representing the volume of the solid obtained by revolving this region about the x-axis.

   Using the cross-section (washer) method, the volume is

$$V = \int_1^e \left(\pi (\ln x)^2 - \pi \left(\frac{x-1}{e-1} \right)^2 \right) dx.$$

11. Consider the solid created by revolving the region bounded by

$$y = x^3, y = 8 \text{ and the } y - \text{axis}$$

about the y-axis. Suppose a tank with this shape is filled with a heavy liquid weighing 90 lb/ft^3. Calculate the work done in pumping all of the liquid to the top of the tank (assume linear units are feet).

A cross section of the solid at y is a disc of radius $y^{\frac{1}{3}}$, so the cross-sectional area is $\pi y^{\frac{2}{3}}$. That cross-section must be lifted a distance of $8 - y$ feet, so the work done in lifting all of the liquid to the top of the tank is

$$W = 90 \int_0^8 \pi y^{\frac{2}{3}} (8-y)\, dy = 90\pi \int_0^8 8y^{\frac{2}{3}} - y^{\frac{5}{3}}\, dy = 90\pi \left(8 \left(\frac{3}{5} y^{\frac{5}{3}} \right) - \frac{3}{8} y^{\frac{8}{3}} \right) \Big|_0^8$$

$$= 90\pi \left(\frac{24}{5} 8^{\frac{5}{3}} - \frac{3}{8} 8^{\frac{8}{3}} \right) = 90\pi \left(\frac{768}{5} - \frac{768}{8} \right) = 5184\pi.$$