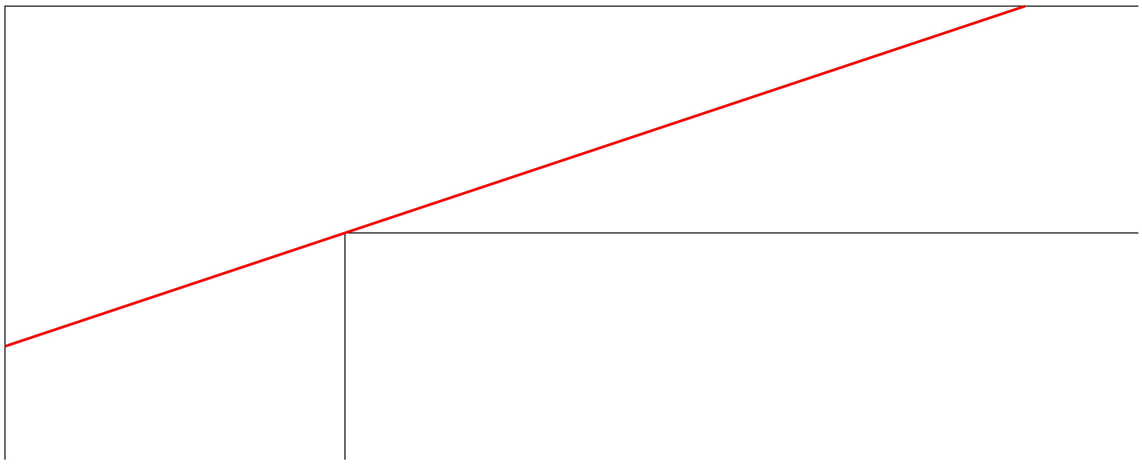
The pipe-around-a-corner or ladder-over-a-fence max/min problem

Suppose there are two hallways that intersect. Suppose we are trying to move a pipe from one hallway to the other, while keeping the pipe horizontal. Given the widths of the hallways, what is the longest pipe we can do this with?

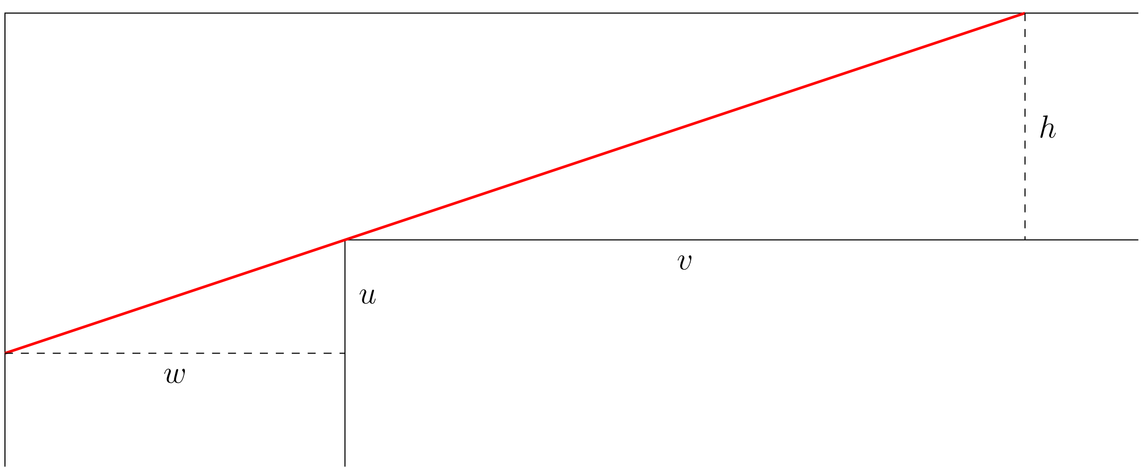
Here is a picture of the situation, viewing the hallways from above.



The red sloping line shows a pipe that someone was trying to get around this corner, but it got stuck. What we are interested in is **the shortest pipe that will get stuck**. If we can figure that out, then we will know that any shorter pipe will make it around the corner.

Let's introduce a few variables to get things started.

Let's say that the hallways have widths w and h, and that the distance short of the corner that the pipe hits the wall is u. Adding in a few dotted lines, and labeling the horizontal distance from the corner to the second time the pipe hits the wall as v, we have the following diagram:



What we would like to do is express the length, L, of the pipe in terms of u. Then we can ask: what u value will give us the smallest possible L? Answering that will answer our question of finding the shortest pipe that will not make it around the corner.

Looking at the diagram, we can use the pythagorean theorem to write

$$L = \sqrt{w^2 + u^2} + \sqrt{v^2 + h^2}.$$

This is great, but it has both u and v in it: we want L in terms of just u.

Using similar triangles, we can write

$$\frac{h}{v} = \frac{u}{w}$$

and so

$$v = \frac{hw}{u}.$$

Thus,

$$\begin{aligned} L &= \sqrt{w^2+u^2} + \sqrt{v^2+h^2} \\ &= \sqrt{w^2+u^2} + \sqrt{\left(\frac{hw}{u}\right)^2 + h^2} \\ &= \sqrt{w^2+u^2} + h\sqrt{\frac{w^2}{u^2}+1}. \end{aligned}$$

Finally, we have L expressed as a function of u. We can now differentiate it with respect to u and find critical points which will tell us what value of u will minimize L:

$$\begin{aligned} \frac{dL}{du} &= \frac{1}{2} \cdot 2u(w^2+u^2)^{-1/2} + h \cdot \frac{1}{2}\left(\frac{w^2}{u^2}+1\right)^{-1/2}\left(\frac{-2w^2}{u^3}\right) \\ &= \frac{u}{\sqrt{w^2+u^2}} - \frac{hw^2}{u^3}\sqrt{\frac{u^2}{w^2+u^2}} \\ &= \frac{u}{\sqrt{w^2+u^2}} - \frac{hw^2}{u^2}\frac{1}{\sqrt{w^2+u^2}} \end{aligned}$$

Setting this equal to zero and clearing denominators, we have

$$u - \frac{hw^2}{u^2} = 0$$

and so

$$u = (hw^2)^{1/3}.$$

How do we know this yields the minimum L? We use the fact that as u goes to zero or infinity, L goes to infinity. Hence, since there is only one critical point, it must be a minimum.

Now here is the good part. What is the length of this shortest-pipe-that-gets-stuck? First, note that we write L as

$$\begin{aligned} L &= \sqrt{w^2+u^2} + h\sqrt{\frac{w^2}{u^2}+1} \\ &= \sqrt{w^2+u^2} + \frac{h}{u}\sqrt{w^2+u^2} \\ &= \left(1+\frac{h}{u}\right)\sqrt{w^2+u^2} \end{aligned}$$

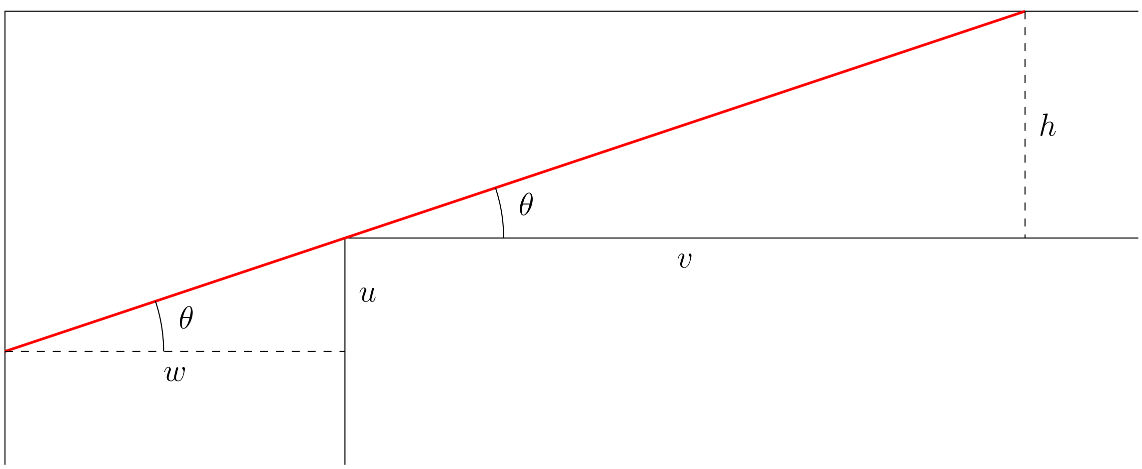
Plugging in $u = (hw^2)^{1/3}$, we get

$$\begin{aligned} L &= \left(1 + \frac{h}{h^{1/3}w^{2/3}}\right)\sqrt{w^2 + h^{2/3}w^{4/3}} \\ &= \left(\frac{w^{2/3} + h^{2/3}}{w^{2/3}}\right) w^{2/3}\sqrt{w^{2/3} + h^{2/3}} \\ &= \left(w^{2/3} + h^{2/3}\right)^{3/2}. \end{aligned}$$

Thus, any pipe shorter than $\left(w^{2/3} + h^{2/3}\right)^{3/2}$ can make it around the corner.

Using trigonometry

By labelling an angle in our original diagram θ, we can pretty quickly get L as a function of θ.



Notice that θ appears twice in the figure.
Working with each triangle in the figure, we can then write

$$L = \frac{w}{\cos\theta} + \frac{h}{\sin\theta}.$$

Differentiating we have

$$\frac{dL}{d\theta} = \frac{w(-\sin\theta)}{\cos^2\theta} - \frac{h\cos\theta}{\sin^2\theta}.$$

Setting this equal to zero and solving, we find

$$w\sin^3\theta - h\cos^3\theta = 0$$

so

$$\tan^3\theta = \frac{h}{w}, \text{ or } \tan\theta = \left(\frac{h}{w}\right)^{1/3}.$$

Utilizing the identities $\tan^2\alpha + 1 = \sec^2\alpha$ and $\sin^2\alpha + \cos^2\alpha = 1$, we find that

$$\cos\theta = \frac{w^{1/3}}{\sqrt{w^{2/3} + h^{2/3}}}$$

and

$$\sin\theta = \frac{h^{1/3}}{\sqrt{w^{2/3} + h^{2/3}}}.$$

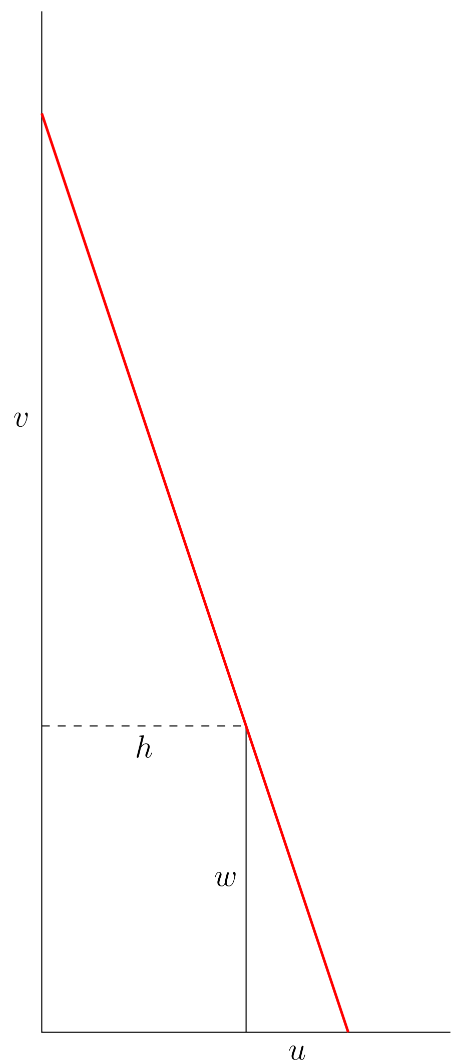
Substituting these into

$$L = \frac{w}{\cos\theta} + \frac{h}{\sin\theta}$$

yields our previous result that the minimum passable length is $\left(w^{2/3} + h^{2/3}\right)^{3/2}$.

Ladder over a fence version

By drawing the right picture (or simply turn our heads 90 degrees), we can see that this is the same as the problem of finding the shortest ladder that can reach a building from behind a fence of a specified height which is a specified distance from the building:



If the fence has height w and is a distance h from the buidling, then the ladder (red line) must have a length of at least

$$(w^{2/3} + h^{2/3})^{3/2}$$

in order to reach the building.