Determining Quadratic Functions

A linear function, of the form $f(x) = ax + b$, is determined by two points. Given two points on the graph of a linear function, we may find the slope of the line which is the function's graph, and then use the point-slope form to write the equation of the line.

A quadratic function, of the form $f(x) = ax^2 + bx + c$, is determined by three points. Given three points on the graph of a quadratic function, we can work out the function by finding a, b and c algebraically.

This will require solving a system of three equations in three unknowns. However, a general solution method is not needed, since the equations all have a certain special form. In particular, they all contain a $+c$ term, and this allows us to simplify to a two variable/two equation system very quickly.

Here is an example.

Suppose we know $f(x) = ax^2 + bx + c$ is a quadratic function and that $f(-2) = 5$, $f(1) = 8$, and $f(6) = 4$. Note this is equivalent to saying that the points $(-2, 5)$, $(1, 8)$ and $(6, 4)$ lie on the graph of f.

These three points give us the following equations:

$$5 = a((-2)^2) + b(-2) + c = 4a - 2b + c \tag{1}$$

$$8 = a(1^2) + b(1) + c = a + b + c \tag{2}$$

$$4 = a(6^2) + b(6) + c = 36a + 6b + c \tag{3}$$

Notice that $+c$ terms dangling on the right-hand end of each equation.

By subtracting the equations in pairs, we eliminate the $+c$ term, and get two equations and two unknowns.

We find, by subtracting equation (2) from equation (1), and by subtracting equation (2) from equation (3), that

$$3a - 3b = -3 \tag{4}$$

$$35a + 5b = -4 \tag{5}$$

This system is then easily solved. We might, for example, simplify equation (4) to

$$b - a = 1$$

so that $b = a + 1$, which when substituted into equation (5), yields

$$35a + 5(a + 1) = -4$$

which gives us

$$a = -\frac{9}{40}$$

from which we determine that

$$b = \frac{31}{40} \text{ and } c = \frac{149}{20}.$$

and so our function is

$$f(x) = -\frac{9}{40}x^2 + \frac{31}{40}x + \frac{149}{20}.$$

This method of subtraction will always work to reduce the system of three equations to a system of two. From that point, any method can be used to solve for a and b, and then one of the original equations is used to find c.