Sinusoidal Function Example (with arcsine)

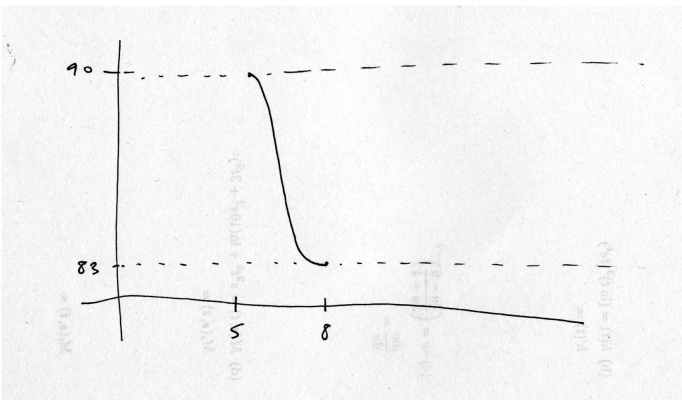
Suppose the temperature of a certain animal is a sinusoidal function of time.

Five hours after you start measuring it, the temperature is at its maximum: 90 degrees Fahrenheit. The temperature than dropped, reaching its minimum of 83 degrees 3 hours later.

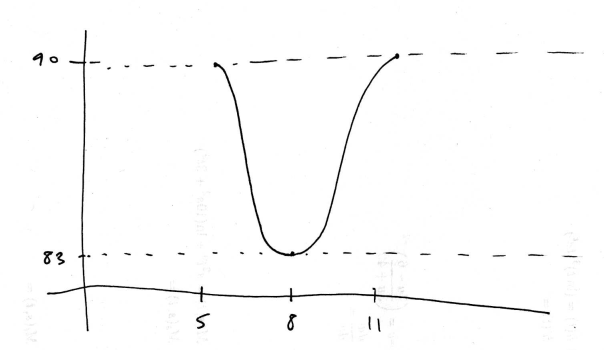
In the first 19 hours during which you are measuring the temperature, how much of the time is the temperature above 84 degrees?

To solve this, we begin by determining the sinusoidal function that gives the temperature in terms of time.

Let $t = 0$ be the time when you start measuring. Then, we sketch a graph of the function, showing as much information as we know:



We then extend the graph so that we see a full period:



Now, we work out the parameters A, B, C and D:

$A = \frac{1}{2}(90 - 83) = 3.5$ $\qquad D = \frac{1}{2}(90 + 83) = 86.5$ $\qquad B = 2(8 - 5) = 6$ $\qquad C = 8 + \frac{B}{4} = 9.5$

So, the function is

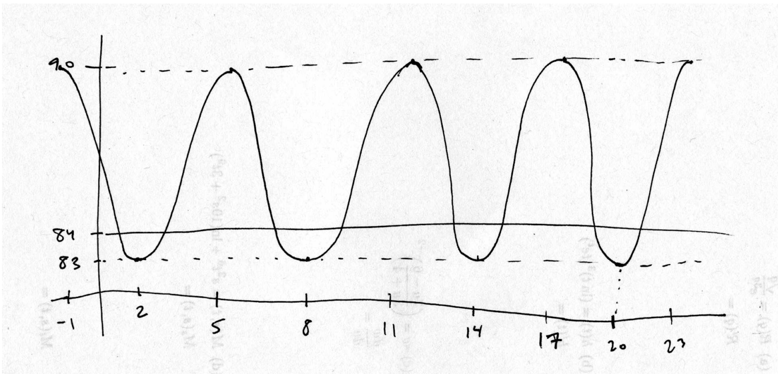
$$f(t) = 3.5 \sin\left(\frac{2\pi}{6}(t - 9.5)\right) + 86.5.$$

Now, since we are interested in how much time the temperature is above 84 degrees, we begin by finding all solutions to the equation

$$f(t) = 84$$

with $0 \le t \le 19$ (since the question is about what happens in the first 19 hours).

Adding periods to our graph so that we cover the interval $0 \le t \le 19$, we have



Here I've added the line $y = 84$ to help our thinking.

Now, we can see there are a number of solutions to find.

We begin by finding one. We use $\sin^{-1}$ to do it:

$$
\begin{aligned}
3.5\sin\left(\frac{2\pi}{6}(t-9.5)\right)+86.5 &= 84 \\
\sin\left(\frac{2\pi}{6}(t-9.5)\right) &= -0.71428571428 \\
\frac{2\pi}{6}(t-9.5) &= \sin^{-1}(-0.71428571428) \\
\frac{2\pi}{6}(t-9.5) &= -0.7956029534845 \\
t &= 8.74025514328654
\end{aligned}
$$

This is our Principal Solution: P = 8.74025514328654. This is the solution nearest to $t = C$, i.e., it will always be in the interval $C - \frac{B}{4} \le t \le C + \frac{B}{4}$.

To find the Symmetry Solution, we utilize the symmetry of the graph of the function. In particular, we use the fact that the graph is symmetric about any vertical line through one of the high points on the graph. The nearest maximum to the right of P is at $t = 11$, so we find our Symmetry Solution like this:

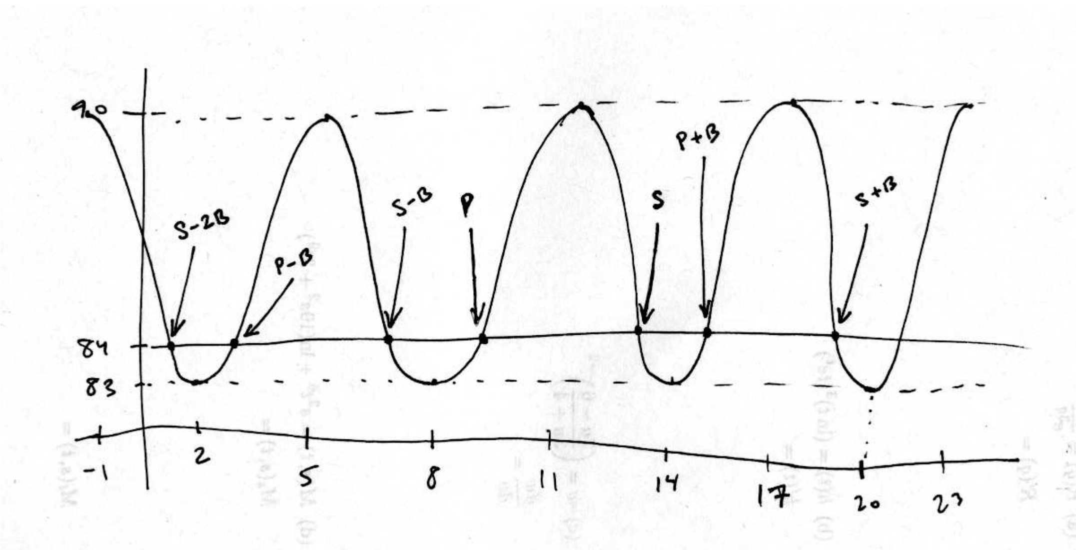
$$S = 11 + (11 - P) = 13.25974485671345041$$

That is, we see S is to the right of the maximum at $t = 11$ (on the "other side of the hill" as it were), so we want to add something on to 11 to get there. What do we add? We add the amount that the Principal solution is to the left of 11: 11-P.

Once we have our Principal Solution, P, and our Symmetry Solution, S, all other solutions are obtained by adding or subtracting the period repeatedly from P and S. We do this to generate all other needed solutions. Here, we want all solutions between 0 and 19. We have:

$$
\begin{aligned}
P + B &= 14.74025514328654 \\
S + B &= 19.25974485671345 \text{ (outside of } 0 \le t \le 19) \\
P - B &= 2.740255143286549 \\
S - B &= 7.259744856713450 \\
P - 2B &= -3.259744856713450 \text{ (outside of } 0 \le t \le 19) \\
S - 2B &= 1.259744856713450 \\
S - 3B &= -4.740255143286549 \text{ (outside of } 0 \le t \le 19)
\end{aligned}
$$

Notating our graph with the t values of the solutions, we have:



We see then that there are four separate intervals of time during which the temperature is above 84 degrees.

The first is from $t = 0$ to $t = S - 2B$: it has length $S - 2B$.

The second is from $t = P - B$ to $t = S - B$: it has length $S - P$.

The third is from $t = P$ to $t = S$: it has length $S - P$.

The fourth is from $t = P + B$ to $t = 19$: it has length $19 - (P + B)$.

All together, the amount of time during which the temperature is above 84 degrees is

$$(S - 2B) + 2(S - P) + (19 - (P + B)) = 14.55846914028070 \text{ hours.}$$

Alternatively, we may note that the time during which the temperature is *below* 84 degrees is

$$3(P - (S - B)) = 4.4415308597192974$$

and so the amount of time above 84 degrees is

$$19 - 4.4415308597192974 = 14.55846914028070.$$