Inverse Function Example

Let's find the inverse function for the function

$$f(x) = \sqrt{x} + 2\sqrt{x+1}.$$

The method is always the same: set $y = f(x)$ and solve for x. If you can get x written as a function of y, then that function is $f^{-1}(y)$.

So, here goes:

$$\begin{aligned}
y &= f(x) \\
y &= \sqrt{x} + 2\sqrt{x+1} \\
y - \sqrt{x} &= 2\sqrt{x+1} \\
(y - \sqrt{x})^2 &= (2\sqrt{x+1})^2 \\
y^2 - 2y\sqrt{x} + x &= 4(x+1) = 4x + 4 \\
y^2 - 2y\sqrt{x} &= 3x + 4 \\
-2y\sqrt{x} &= (3x+4) - y^2 \\
4y^2 x &= (3x+4)^2 - 2y^2(3x+4) + y^4 \\
4y^2 x &= 9x^2 + 24x + 16 - 6y^2 x - 8y^2 + y^4
\end{aligned}$$

So, finally, we have

$$0 = 9x^2 + (24 - 10y^2)x + 16 - 8y^2 + y^4$$

We can use the quadratic formula to solve for x:

$$\begin{aligned}
x &= \frac{10y^2 - 24 \pm \sqrt{(24 - 10y^2)^2 - 36(16 - 8y^2 + y^4)}}{18} \\
&= \frac{5}{9}y^2 - \frac{4}{3} \pm \frac{1}{18}\sqrt{64y^4 - 192y^2} \\
&= \frac{5}{9}y^2 - \frac{4}{3} \pm \frac{4}{9}\sqrt{y^4 - 3y^2}.
\end{aligned}$$

Thus, we have, at last *almost* found an inverse for $f(x)$. Two bits of trouble: (1) the $\pm$ business suggests that perhaps $f(x)$ is not one-to-one, and (2) if it is one-to-one, which of the $+$ or $-$ do we pick?

Here we will rely on our knowledge of the square root function, $\sqrt{x}$. It is always increasing: that is, if $a > b$ then $\sqrt{a} > \sqrt{b}$. This means that the graph of $\sqrt{x}$ moves upward as we move from left to right. From this you can conclude that $\sqrt{x}$ is one-to-one. Also, since $\sqrt{x+1}$ is a horizontal shift of $\sqrt{x}$, it is also an increasing function. Multiplying it by two does not change that, so $2\sqrt{x+1}$ is an increasing function too. Finally, if you add two increasing functions together, you get an increasing function. So, $f(x) = \sqrt{x} + 2\sqrt{x+1}$ is an increasing function, and is one-to-one.

With that taken care of, now we just have to decide what to do with the $\pm$. One way to deal with it is by checking a point (or more if you like) to see which one actually undoes $f(x)$.

For instance,

$$f(4) = \sqrt{4} + 2\sqrt{5} = 6.47213595....$$

Plugging this in for y in

$$x = \frac{5}{9}y^2 - \frac{4}{3} \pm \frac{4}{9}\sqrt{y^4 - 3y^2}$$

gives us

$$x = 21.93807989999906531 \pm 17.93807989999906531$$

Clearly, we'll have to take the $-$ option to get $x = 4$. Thus,

$$f^{-1}(y) = \frac{5}{9}y^2 - \frac{4}{3} - \frac{4}{9}\sqrt{y^4 - 3y^2}$$

or, if you like

$$f^{-1}(x) = \frac{5}{9}x^2 - \frac{4}{3} - \frac{4}{9}\sqrt{x^4 - 3x^2}$$

Whew.

Here is a figure showing the function, $f(x)$ (the solid curve) and its inverse function $f^{-1}(x)$ (the dashed curve). The line $y = x$ is shown to so you can clearly see that the graphs are symmetric with respect to that line. An inverse function will always have a graph that looks like a mirror image of the original function, with the line $y = x$ as the mirror.