Inverse Function Example

Let’s find the inverse function for the function

\[ f(x) = \sqrt{x} + 2\sqrt{x+1}. \]

The method is always the same: set \( y = f(x) \) and solve for \( x \). If you can get \( x \) written as a function of \( y \), then that function is \( f^{-1}(y) \).

So, here goes:

\[
\begin{align*}
y &= f(x) \\
y &= \sqrt{x} + 2\sqrt{x+1} \\
y - \sqrt{x} &= 2\sqrt{x+1} \\
(y - \sqrt{x})^2 &= (2\sqrt{x+1})^2 \\
y^2 - 2y\sqrt{x} + x &= 4(x+1) = 4x + 4 \\
y^2 - 2y\sqrt{x} &= 3x + 4 \\
-2y\sqrt{x} &= (3x + 4) - y^2 \\
4y^2x &= (3x + 4)^2 - 2y^2(3x + 4) + y^4 \\
4y^2x &= 9x^2 + 24x + 16 - 6y^2x - 8y^2 + y^4
\end{align*}
\]

So, finally, we have

\[ 0 = 9x^2 + (24 - 10y^2)x + 16 - 8y^2 + y^4 \]

We can use the quadratic formula to solve for \( x \):

\[
x = \frac{10y^2 - 24 \pm \sqrt{(24 - 10y^2)^2 - 36(16 - 8y^2 + y^4)}}{18}
\]

\[
= \frac{5y^2 - 4}{3} \pm \frac{1}{18} \sqrt{64y^4 - 192y^2}
\]

\[
= \frac{5y^2 - 4}{3} \pm \frac{4}{9} \sqrt{y^4 - 3y^2}
\]

Thus, we have, at last almost found an inverse for \( f(x) \). Two bits of trouble: (1) the \( \pm \) business suggests that perhaps \( f(x) \) is not one-to-one, and (2) if it is one-to-one, which of the + or − do we pick?

Here we will rely on our knowledge of the square root function, \( \sqrt{x} \). It is always increasing: that is, if \( a > b \) then \( \sqrt{a} > \sqrt{b} \). This means that the graph of \( \sqrt{x} \) moves upward as we move from left to right. From this you can conclude that \( \sqrt{x} \) is one-to-one. Also, since \( \sqrt{x+1} \) is a horizontal shift of \( \sqrt{x} \), it is also an increasing function. Multiplying it by two does not change that, so \( 2\sqrt{x+1} \) is an increasing function too. Finally, if you add two increasing functions together, you get an increasing function. So, \( f(x) = \sqrt{x} + 2\sqrt{x+1} \) is an increasing function, and is one-to-one.

With that taken care of, now we just have to decide what to do with the \( \pm \). One way to deal with it is by checking a point (or more if you like) to see which one actually undoes \( f(x) \).

For instance,

\[ f(4) = \sqrt{4} + 2\sqrt{5} = 6.47213595.... \]

Plugging this in for \( y \) in

\[ x = \frac{5y^2 - 4}{3} \pm \frac{4}{9} \sqrt{y^4 - 3y^2} \]
gives us
\[ x = 21.93807989999906531 \pm 17.93807989999906531 \]
Clearly, we’ll have to take the – option to get \( x = 4 \). Thus,
\[
f^{-1}(y) = \frac{5}{9}y^2 - \frac{4}{3} - \frac{4}{9} \sqrt{y^4 - 3y^2}
\]
or, if you like
\[
f^{-1}(x) = \frac{5}{9}x^2 - \frac{4}{3} - \frac{4}{9} \sqrt{x^4 - 3x^2}
\]
Whew.
Here is a figure showing the function, \( f(x) \) (the solid curve) and its inverse function \( f^{-1}(x) \) (the dashed curve). The line \( y = x \) is shown so you can clearly see that the graphs are symmetric with respect to that line. An inverse function will always have a graph that looks like a mirror image of the original function, with the line \( y = x \) as the mirror.